Tighter Uncertainty and Reverse Uncertainty Relations

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Plan

- Introduction.

Plan

- Introduction.
- Uncertainty relations

Plan

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- Uncertainty relations
- Tighter uncertainty relations

Plan

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- Uncertainty relations
- Tighter uncertainty relations
- Reverse uncertainty relations

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- Introduction.
- Uncertainty relations
- Tighter uncertainty relations
- Reverse uncertainty relations
- Summary and conclusions

History

- Quantum mechanics is the finest theory of nature that explains behavior of atoms and subatomic particles.
- It has been enormously successful in giving correct results in practically every situation.
- In spite of the overwhelming success of quantum mechanics, the foundations of the subject contain many paradoxes.

History

- (Heisenberg) Matrix mechanics [Zeitschrift für Physik, **33**, 879-893 (1925)].
- (Schrödinger) wave mechanics. [E. Schrödinger, Ann. der Physik, **384**, 361-376 (1926)].
- The wave mechanics appealed to many physicists. It is equivalent to matrix mechanics.
- Intense debate between the alternative versions of quantum mechanics formed the background for the development of uncertainty relation and the Copenhagen Interpretation.

History

- I knew of [Heisenberg's] theory, of course, but I felt discouraged, not to say repelled, by the methods of transcendental algebra, which appeared difficult to me, and by the lack of visualizability.
  -Schrödinger in 1926

- The more I think about the physical portion of Schödinger's theory, the more repulsive I find it...What Schrödinger writes about the visualizability of his theory 'is probably not quite right,' in other words it's crap.
  –Heisenberg, writing to Pauli (1926)

History

- It is meaningless to ascribe any properties or even existence to anything that has not been measured.

  Bohr: “Nothing is real unless it is observed”.
- Einstein: It is the theory which decides what we can observe.
- I believe that the existence of the classical “path” can be formulated as follows: The ”path” comes into existence only when we observe it.
  –Heisenberg, in uncertainty principle paper (1927)

Uncertainty Principle

- (1927) Heisenberg's uncertainty principle is a fundamental result in quantum physics and gives profound insights to quantum world.
- Minimum amount of unavoidable momentum disturbance and inaccuracy in position measurement $\Delta x \Delta p \sim h$
- However, he did not give a precise definition for Δx and Δp.

  One can never know with perfect accuracy both of those two important factors which determine the movement of one of the smallest particle–its position and its velocity.

Uncertainty Relation

- Robertson Uncertainty Relation (1927): Bounds the product of the variances for two observables through the expectation value of the commutator

$$\Delta A^2 \Delta B^2 \geq |\frac{1}{2}\langle\psi|[A,B]|\psi\rangle|^2 \quad (1)$$

- There is a limit to the precision with which a pair of non-commuting observables can be measured.

- Robertson and Kennard (1927) relation $\Delta x \Delta p \geq \frac{\hbar}{2}$.

  More precisely the position of a particle is determined, the less precisely its momentum can be known.

Uncertainty Principle vs Relation

- Uncertainty principle is different than uncertainty relation. Both appear in similar formulations that even many practicing physicists tend to confuse.
- Heisenberg version is about uncertainty principle: in observing a quantum particle we inevitably disturb it.
- When we measure the position of an electron with an error Δx, we disturb the momentum of the electron by the amount Δp.

- This (UR) inequality shows that the fluctuation exists regardless whether we measur or not–preparation uncertainty.
- This inequality does not say anything about what happens when a measurement is performed.
- Robertson-Kennard formulation is therefore totally different from Heisenberg's.
- But many physicists, probably including Heisenberg himself, have been under the impression that both formulations describe same phenomenon.

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- Recently, along with Maccone we have discovered stronger uncertainty relations for all incompatible observables. The stronger uncertainty relations have also been experimentally tested.
- Entropic uncertainty relations also capture the essence of quantum uncertainty and the incompatibility between two observables, but in a state-independent way.

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- In most of these areas, particularly, in quantum entanglement detection and quantum metrology or quantum speed limit, where a small fluctuation in an unknown parameter of the state of the system is needed to detect, state-dependent relations may be useful.
- Thus, a focus on the study of the state dependent, tighter uncertainty and the reverse uncertainty relations based on the variance is important.

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- It may be noted that earlier uncertainty relations that provide a bound to the sum of the variances comprise a lower bound in terms of the variance of the sum of observables, entropic uncertainty relations, sum uncertainty relation for angular momentum observables, sum uncertainty relations for N-incompatible observables, uncertainty relations for noise and disturbance and also uncertainty relations for non-Hermitian operators.

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- Recently, experiments have also been performed to test various uncertainty relations.

Uncertainty relations

- For any two non-commuting operators A and B, the Robertson-Schrödinger uncertainty relation for the state of the system $|\Psi\rangle$ is given by the following inequality

$$\Delta A^2 \Delta B^2 \geq \left| \frac{1}{2} \langle [A, B] \rangle \right|^2 + \left| \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right|^2, \tag{2}$$

  where the averages and the variances are defined over the state of the system $|\Psi\rangle$. This relation is a direct consequence of the Cauchy-Schwarz inequality. However, this uncertainty bound is not optimal.

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  Let us consider two observables A and B in their eigenbasis as $A = \sum_i a_i |a_i\rangle\langle a_i|$ and $B = \sum_i b_i |b_i\rangle\langle b_i|$. Let us define $(A - \langle A \rangle) = \overline{A} = \sum_i \tilde{a}_i |a_i\rangle\langle a_i|$ and $(B - \langle B \rangle) = \overline{B} = \sum_i \tilde{b}_i |b_i\rangle\langle b_i|$. We express $|f\rangle = \overline{A}|\Psi\rangle$ and $|g\rangle = \overline{B}|\Psi\rangle$ as $|f\rangle = \sum_n \alpha_n |\psi_n\rangle$ and $|g\rangle = \sum_n \beta_n |\psi_n\rangle$, where $\{|\psi_n\rangle\}$ is an arbitrary complete orthonormal basis.

Tighter uncertainty relations

- Using the Cauchy-Schwarz inequality for two real vectors $\vec{\alpha} = \Big(|\alpha_1|, |\alpha_2|, |\alpha_3|, ...\Big)$, $\vec{\beta} = \Big(|\beta_1|, |\beta_2|, |\beta_3|, ...\Big)$, we have

$$\begin{aligned} \Delta A^2 \Delta B^2 &= \langle f|f\rangle\langle g|g\rangle = \sum_{n,m} |\alpha_n|^2 |\beta_m|^2 \\ &\geq \Big(\sum_n |\alpha_n||\beta_n|\Big)^2 = \Big(\sum_n |\alpha_n^* \beta_n|\Big)^2 \\ &= \Big(\sum_n |\langle\Psi|\overline{A}|\psi_n\rangle\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2 \\ &= \Big(\sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2, \end{aligned} \tag{3}$$

where $\overline{B}_n^{\psi} = |\psi_n\rangle\langle\psi_n|\overline{B}$, $\alpha_n = \langle\psi_n|\overline{A}|\Psi\rangle$ and $\beta_n = \langle\psi_n|\overline{B}|\Psi\rangle$.

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&\geq \Big(\sum_n |\alpha_n||\beta_n|\Big)^2 = \Big(\sum_n |\alpha_n^* \beta_n|\Big)^2 \\
&= \Big(\sum_n |\langle\Psi|\overline{A}|\psi_n\rangle\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2 \\
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- On expressing $\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle = \frac{1}{2}\big(\langle[\overline{A}, \overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A}, \overline{B}_n^{\psi}\}\rangle_\Psi\big)$, the new uncertainty relation can be written as

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}\Big(\sum_n \Big|\langle[\overline{A}, \overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A}, \overline{B}_n^{\psi}\}\rangle_\Psi\Big|\Big)^2. \qquad (4)$$

Tighter uncertainty relations

- The new uncertainty relation is tighter than the Robertson-Schrödinger uncertainty relation. To prove this let us start with the right hand side of Eq. (3) and note that

$$\begin{aligned}\Big(\quad \sum_n |\langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle|\Big)^2 &\geq \Big|\sum_n \langle\Psi|\overline{A}\,\overline{B}_n^{\psi}|\Psi\rangle\Big|^2 \\ = \quad & \Big|\langle\Psi|\overline{A}\,\overline{B}|\Psi\rangle\Big|^2, \end{aligned} \tag{5}$$

where we have used the fact that $|\sum_i z_i|^2 \leq (\sum_i |z_i|)^2$, $z_i \in \mathbb{C}$ for all i. Here, the last line in Eq. (5) is nothing but the bound obtained in Eq. (2).

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- Thus, our bound is indeed tighter than the Robertson-Schrödinger uncertainty relation.
- This uncertainty relation in Eq. (4) can further be tightened by optimizing over the sets of complete orthonormal bases as

$$\Delta A^2 \Delta B^2 \geq \max_{\{|\psi_n\rangle\}} \frac{1}{4}\Big(\sum_n \Big|\langle[\overline{A}, \overline{B}_n^{\psi}]\rangle_\Psi + \langle\{\overline{A}, \overline{B}_n^{\psi}\}\rangle_\Psi\Big|\Big)^2. \tag{6}$$

Tighter uncertainty relations

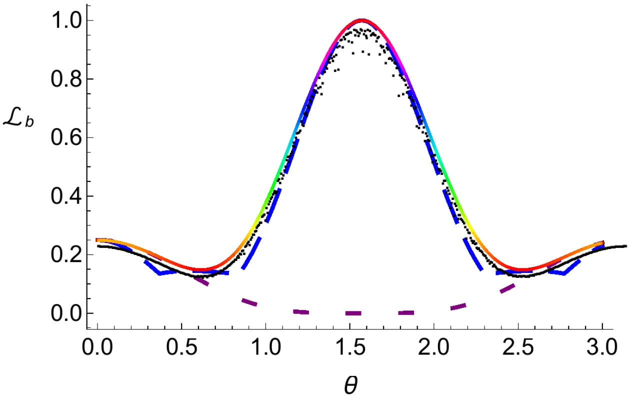

Figure : Here, we plot the lower bound of the product of variances of two incompatible observables, $A = L_x$ and $B = L_y$, two components of the angular momentum for spin 1 particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the state $|1\rangle$ and $|0\rangle$ are the eigenstates of L_z corresponding to eigenvalues 1 and 0 respectively. The long dashed (blue colored) line shows the lower bound of the product of variances given by (7), the flattest (purple colored, tiny dashed) curve stands for the bound given by Schrödinger uncertainty relation given by Eq. (2) and the continuous line (hue colored) plot denotes the product of two variances. Scattered black points denote the optimized uncertainty bound achieved by Eq. (6).

Tighter uncertainty relations

- Next, we derive an optimization-free uncertainty relation for two incompatible observables. For that we consider (say) $\overline{A}^2 = \sum_{i,j}(a_i - a_j F_{\Psi}^{a_j})^2 |a_i\rangle\langle a_i| = \sum_i (\tilde{a}_i)^2 |a_i\rangle\langle a_i|$ and $\overline{B}^2 = \sum_{i,j}(b_i - b_j F_{\Psi}^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_{Ψ}^{x} is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ $(|x\rangle = |a_i\rangle, |b_i\rangle)$, $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.

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- Using the Cauchy-Schwarz inequality, we obtain

$$\Delta A^2 \Delta B^2 \geq \left(\sum_i \sqrt{F_{\Psi}^{a_i}} \sqrt{F_{\Psi}^{b_i}} \tilde{a}_i \tilde{b}_i \right)^2, \quad (7)$$

where we use the inequality for two real vectors $\vec{u}$ and $\vec{v}$ defined as $\vec{u} = \left(\tilde{a}_1 \sqrt{F_{\Psi}^{a_1}}, \tilde{a}_2 \sqrt{F_{\Psi}^{a_2}}, \tilde{a}_3 \sqrt{F_{\Psi}^{a_3}}, \ldots \right)$, $\vec{v} = \left(\tilde{b}_1 \sqrt{F_{\Psi}^{b_1}}, \tilde{b}_2 \sqrt{F_{\Psi}^{b_2}}, \tilde{b}_3 \sqrt{F_{\Psi}^{b_3}}, \ldots \right)$ and the quantities $\sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$, $\sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$ are arranged such that $\sqrt{F_{\Psi}^{a_{i+1}}}\tilde{a}_{i+1} \geq \sqrt{F_{\Psi}^{a_i}}\tilde{a}_i$ and $\sqrt{F_{\Psi}^{b_{i+1}}}\tilde{b}_{i+1} \geq \sqrt{F_{\Psi}^{b_i}}\tilde{b}_i$.

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  $\overline{B}^2 = \sum_{i,j}(b_i - b_j F_\Psi^{b_j})^2 |b_i\rangle\langle b_i| = \sum_i (\tilde{b}_i)^2 |b_i\rangle\langle b_i|$, where F_Ψ^x is nothing but the fidelity between the state $|\Psi\rangle$ and $|x\rangle$ ($|x\rangle = |a_i\rangle, |b_i\rangle$), $F(|\Psi\rangle, |x\rangle) = |\langle\Psi|x\rangle|^2$.
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- This new uncertainty relation depends on the transition probability between the state of the system and the eigenbases of the observables. The incompatibility is captured here not by the non-commutativity, rather by the non-orthogonality of the state of the system $|\Psi\rangle$ and the eigenbases of the observables $|a_i\rangle$ and $|b_i\rangle$.

Tighter uncertainty relations

- However, we know that the product of variances does not fully capture the uncertainty for two incompatible observables, since if the state of the system is an eigenstate of one of the observables, then the product of the uncertainties vanishes. To overcome this shortcoming, the sum of variances was invoked to capture the uncertainty of two incompatible observables. In this regard, stronger uncertainty relations for all incompatible observables were proposed. But, these uncertainty relations are not always tight and highly dependent on the states perpendicular to the chosen state of the system. Here, we propose new uncertainty relations that perform better than the existing bounds and need no optimization. We use the the parallelogram law for two real vectors to improve the bound on the sum of variances for two incompatible observables.

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- Using the parallelogram law for two real vectors $\vec{u}$ and $\vec{v}$, one can derive a lower bound on the sum of variances of two observables as

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2} \sum_i \left(\tilde{a}_i \sqrt{F_\Psi^{a_i}} + \tilde{b}_i \sqrt{F_\Psi^{b_i}} \right)^2 . \quad (8)$$

As shown in Fig. (2), the bound obtained in Eq. (8) is one of the tightest optimization free bound.

Tighter uncertainty relations

- However, we know that the product of variances does not fully capture the uncertainty for two incompatible observables, since if the state of the system is an eigenstate of one of the observables, then the product of the uncertainties vanishes. To overcome this shortcoming, the sum of variances was invoked to capture the uncertainty of two incompatible observables. In this regard, stronger uncertainty relations for all incompatible observables were proposed. But, these uncertainty relations are not always tight and highly dependent on the states perpendicular to the chosen state of the system. Here, we propose new uncertainty relations that perform better than the existing bounds and need no optimization. We use the the parallelogram law for two real vectors to improve the bound on the sum of variances for two incompatible observables.
- Using the parallelogram law for two real vectors $\vec{u}$ and $\vec{v}$, one can derive a lower bound on the sum of variances of two observables as

$$\Delta A^2 + \Delta B^2 \quad \geq \quad \frac{1}{2} \sum_i \left(\tilde{a}_i \sqrt{F_\Psi^{a_i}} + \tilde{b}_i \sqrt{F_\Psi^{b_i}} \right)^2 . \tag{8}$$

  As shown in Fig. (2), the bound obtained in Eq. (8) is one of the tightest optimization free bound.
- If one allows the optimization over a set of states, then the procedure used to derive the uncertainty relation given in Eq. (6) can be used to derive another set of uncertainty relations using the parallelogram law for two real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Tighter uncertainty relations

Using the parallelogram law, one obtains

$$\begin{aligned} \Delta A^2 + \Delta B^2 &\geq \frac{1}{2}\sum_n \Big(|\alpha_n| + |\beta_n|\Big)^2 \\ &= \frac{1}{2}\sum_n \Big(|\langle\psi_n|\overline{A}|\Psi\rangle| + |\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2. \end{aligned} \quad (9)$$

An optimization over the set of complete bases provides a more tighter bound as

$$\Delta A^2 + \Delta B^2 \geq \max_{\{|\psi_n\rangle\}} \frac{1}{2}\sum_n \Big(|\langle\psi_n|\overline{A}|\Psi\rangle| + |\langle\psi_n|\overline{B}|\Psi\rangle|\Big)^2. \quad (10)$$

Tighter uncertainty relations

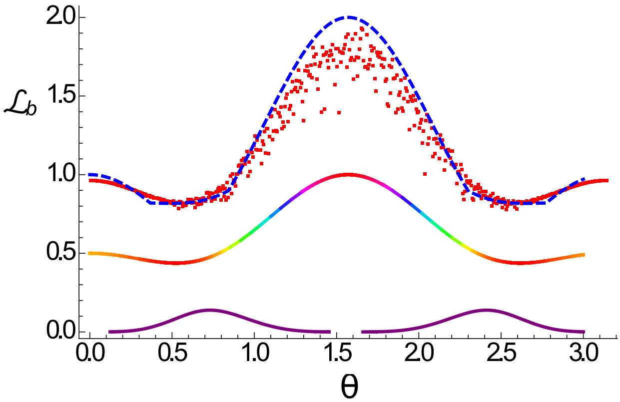

Figure : Here, we plot the lower bound of the sum of variances for two incompatible observables, $A = L_x$ and $B = L_y$, two components of the angular momentum for spin-1 particle with a state $|\Psi\rangle = \cos\theta|1\rangle - \sin\theta|0\rangle$, where the state $|1\rangle$ and $|0\rangle$ are the eigenstates of L_z corresponding to eigenvalues 1 and 0 respectively. Blue dashed line curve shows the lower bound of the sum of variances given by (8), the continuous line (hue colored) plot denotes the bound given by Eq. (4) in and the flattest discontinuous (purple coloured) plot gives the bound given by Eq. (2). Scattered red points are the uncertainty bound achieved by Eq. (3). As observed from the plot, the bound given by Eq. (8) is one of the tightest bounds in the literature. The bound given by Eq. (3) in is the only bound, which surpasses at only few points.

Reverse uncertainty relations

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- This states that *for two sets of positive real numbers* $c_1, ..., c_n$ *and* $d_1, ...d_n$*, if* $0 < c \leq c_i \leq C < \infty$, $0 < d \leq d_i \leq D < \infty$ *for some constants* c, d, C *and* D *for all* $i = 1, ...n$*, then*

$$\sum_{i,j} c_i^2 d_j^2 \leq \frac{(CD + cd)^2}{4cdCD} \Big(\sum_i c_i d_i \Big)^2. \tag{11}$$

Reverse uncertainty relations

- Using this inequality for $c_i = \sqrt{F_\Psi^{a_i}}|\tilde{a}_i|$ and $d_i = \sqrt{F_\Psi^{b_i}}|\tilde{b}_i|$, one can show that the product of variances of two observables satisfies the relation

$$\Delta A^2 \Delta B^2 \leq \Omega_{ab}^{\Psi} \left(\sum_i \sqrt{F_\Psi^{a_i}} \sqrt{F_\Psi^{b_i}} |\tilde{a}_i||\tilde{b}_i| \right)^2, \quad (12)$$

where $\Omega_{ab}^{\Psi} = \frac{\left(M_\Psi^a M_\Psi^b + m_\Psi^a m_\Psi^b\right)^2}{4 M_\Psi^a M_\Psi^b m_\Psi^a m_\Psi^b}$ with $M_\Psi^a = \max\{\sqrt{F_\Psi^{a_i}}|\tilde{a}_i|\}$, $m_\Psi^a = \min\{\sqrt{F_\Psi^{a_i}}|\tilde{a}_i|\}$, $M_\Psi^b = \max\{\sqrt{F_\Psi^{b_i}}|\tilde{b}_i|\}$ and $m_\Psi^b = \min\{\sqrt{F_\Psi^{b_i}}|\tilde{b}_i|\}$.

- If one uses the reverse Cauchy-Schwarz inequality for the two real positive vectors $\vec{\alpha}$ and $\vec{\beta}$, we have

$$\begin{aligned} \Delta A^2 \quad \Delta B^2 &\leq \Lambda_{\alpha\beta}^{\psi\Psi} \left(\sum_n |\alpha_n||\beta_n| \right)^2 \\ &= \frac{\Lambda_{\alpha\beta}^{\psi\Psi}}{4} \left(\sum_n \left| \langle [\overline{A}, \overline{B}_n^{\psi}] \rangle + \langle \{\overline{A}, \overline{B}_n^{\psi}\} \rangle \right| \right)^2, \end{aligned} \quad (13)$$

where $\Lambda_{\alpha\beta}^{\psi\Psi} = \frac{\left(M_{\psi\Psi}^{\alpha} M_{\psi\Psi}^{\beta} + m_{\psi\Psi}^{\alpha} m_{\psi\Psi}^{\beta}\right)^2}{4 M_{\psi\Psi}^{\alpha} M_{\psi\Psi}^{\beta} m_{\psi\Psi}^{\alpha} m_{\psi\Psi}^{\beta}}$ with $M_{\psi\Psi}^{\alpha} = \max\{|\alpha_n|\}$, $m_{\psi\Psi}^{\alpha} = \min\{|\alpha_n|\}$, $M_{\psi\Psi}^{\beta} = \max\{|\beta_n|\}$ and $m_{\psi\Psi}^{\beta} = \min\{|\beta_n|\}$.

- One can optimize the right hand side.

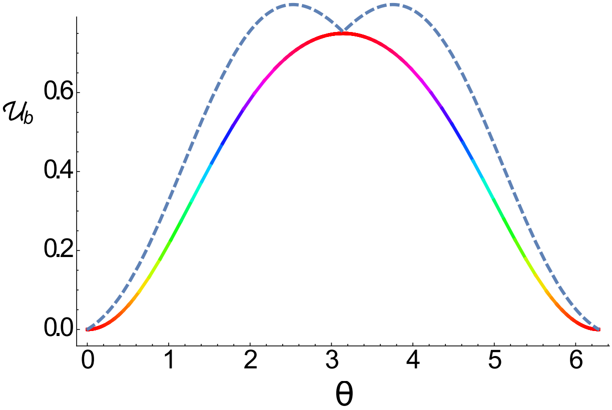

Figure : Here, we plot the upper bound of the product of variances for two incompatible observables, $A = \sigma_x$ and $B = \sigma_z$, two components of the angular momentum for spin $\frac{1}{2}$ particle with a state $\rho = \frac{1}{2}\left(I_2 + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_y + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right)$. Blue dashed line curve is the upper bound of the product of the two variances given by (12) and the continuous line (hue colored) plot denotes the product of the two variances.

Reverse uncertainty relations

- Next, we derive the reverse uncertainty relation for the sum of variances using the Dunkl-Williams inequality. It is a state dependent upper bound on the sum of variances. The Dunkl-Williams inequality states that *if f, g are non-null vectors in the real or complex inner product space, then*

$$||f-g|| \geq \frac{1}{2}(||f||+||g||)||\frac{f}{||f||}-\frac{g}{||g||}||. \tag{14}$$

- Now, if we take $|f\rangle = \overline{A}|\Psi\rangle$ and $|g\rangle = \overline{B}|\Psi\rangle$ as defined earlier, then, using the Dunkl-Williams inequality we obtain the following equation

$$\Delta A + \Delta B \leq \frac{\sqrt{2}\Delta(A-B)}{\sqrt{1-\frac{Cov(A,B)}{\Delta A.\Delta B}}}, \tag{15}$$

where, $Cov(A,B) = \frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle$ is the *quantum covariance* of the operators A and B in quantum state $|\Psi\rangle$. Since $\frac{Cov(A,B)}{\Delta A \Delta B} \leq 1$ the quantity inside the square root in the denominator of the Eq. (15) is always positive.

• Squaring the both sides of the equation, we obtain an upper bound on the sum of variances as

$$\Delta A^2 + \Delta B^2 \leq \frac{2\Delta(A-B)^2}{\left[1-\frac{Cov(A,B)}{\Delta A.\Delta B}\right]} - 2\Delta A\Delta B. \tag{16}$$

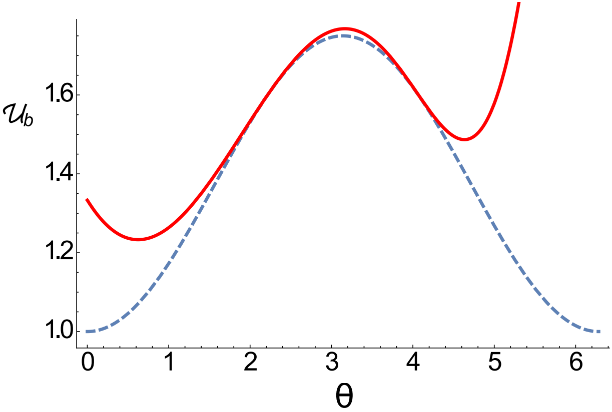

Figure : Here, we plot the upper bound of the sum of variances for two incompatible observables, $A = \sigma_x$ and $B = \sigma_z$, two components of the spin angular momentum for spin $\frac{1}{2}$ particle with a state $\rho = \frac{1}{2}\left(I_2 + \cos\frac{\theta}{2}\sigma_x + \frac{\sqrt{3}}{2}\sin\frac{\theta}{2}\sigma_y + \frac{1}{2}\sin\frac{\theta}{2}\sigma_z\right)$. The continuous line (red colored) plot is the upper bound of the sum of the two variances given by (16) and the blue dashed curve denotes the sum of the two variances.

As can be seen in Fig. (4), the bound is actually tight for some classes of qubit states.

Summary and conclusions

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- The reverse uncertainty relations should set the stage for addressing an important issue in quantum metrology, i.e., to set the upper bound of error in measurement and the upper bound for the time of quantum evolutions.

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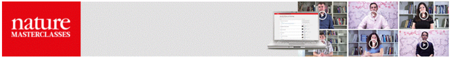

RESEARCH HIGHLIGHTS

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Quantum physics gets a reverse uncertainty relation

Theoretical physicists from the Harish-Chandra Research Institute (HRI) in Allahabad have derived a new kind of relation in quantum mechanics called the "Reverse Uncertainty Relation", that could find applications in various areas of quantum physics, quantum information and quantum technology[1].

Debasis Mondal, Shrobona Bagchi and Arun Kumar Pati from HRI show, for the first time, that there is an upper limit to how accurately one can simultaneously measure the position and momentum of a particle.

The original uncertainty principle introduced in 1927 by Werner Heisenberg is a rule in quantum mechanics which sets a "lower" limit on the product of the "variances" of two "incompatible observables" (such as position and momentum), but it was not known if there is any "upper" limit.

"We show that there is indeed an upper limit," Pati, one of the authors, told *Nature India*. "The reverse uncertainty relation shows that there is a "spread" or "range" for both the sum and product of variances of two non-commuting observables," he said. In addition to the reverse uncertainty relation, the authors have proved a new and tighter uncertainty relation from which the Heisenberg uncertainty relation directly follows.

The new relation may be useful in setting an upper limit in "quantum metrology," which exploits quantum systems to reach unprecedented levels of precision in measurements. "Thus, this is not only of fundamental interest but can have applications in diverse areas of quantum physics," Pati said adding "the reverse uncertainty relation should open up a whole new direction of explorations in quantum mechanics which we have not thought of."

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THANK YOU