Sharing of nonlocality

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Plan

- Introduction: Definition of the problem

- Information gain versus disturbance trade-off --- Optimality for POVM

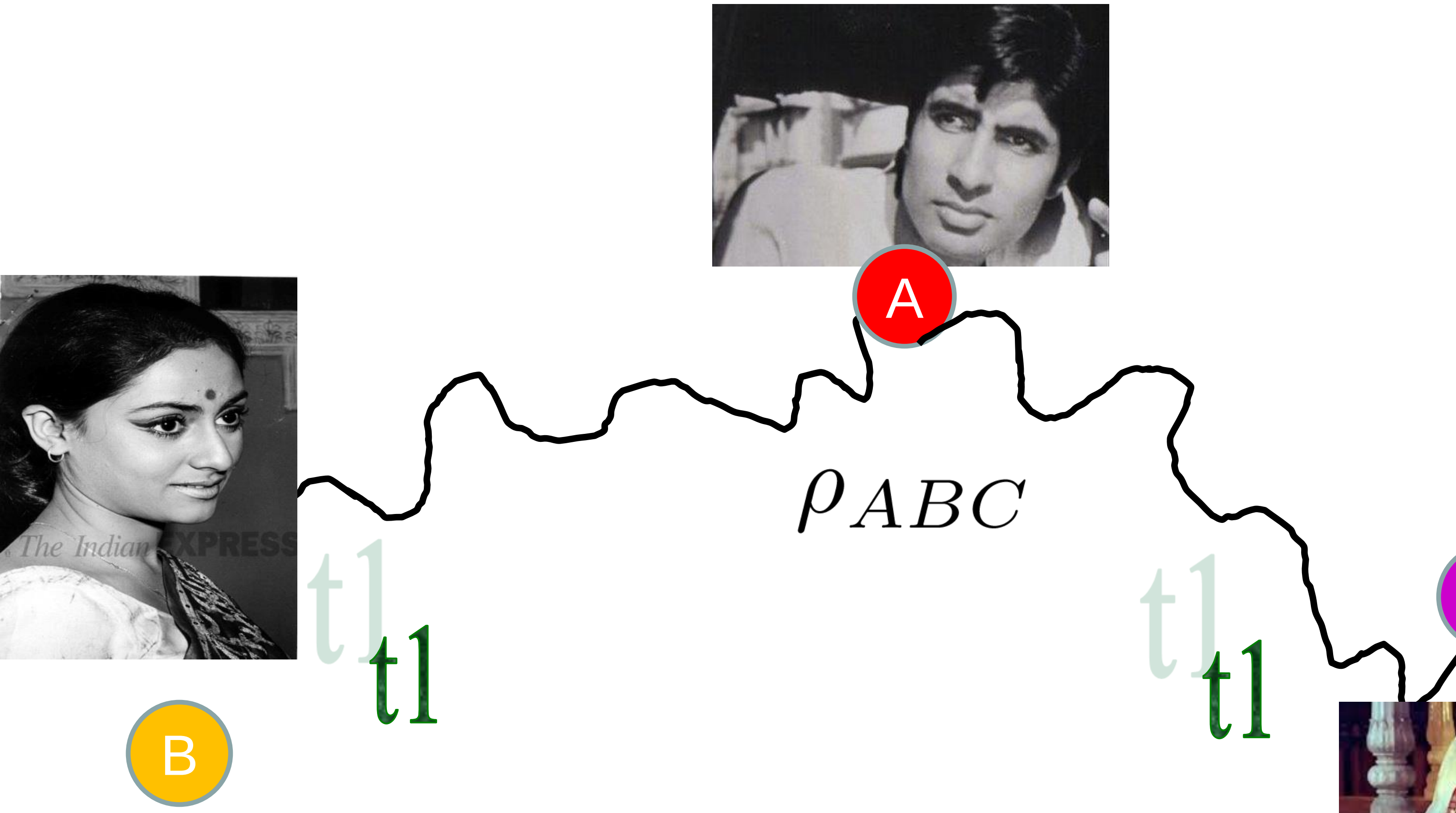
- Proof of impossibility of CHSH violation by Alice and more than two Bobs acting sequentially and independently

- Violation of 3-settings steering inequality by Alice and three Bobs

What do we mean by sharing of nonlocality ?

Let us first consider sharing of entanglement

Entanglement is Monogamous

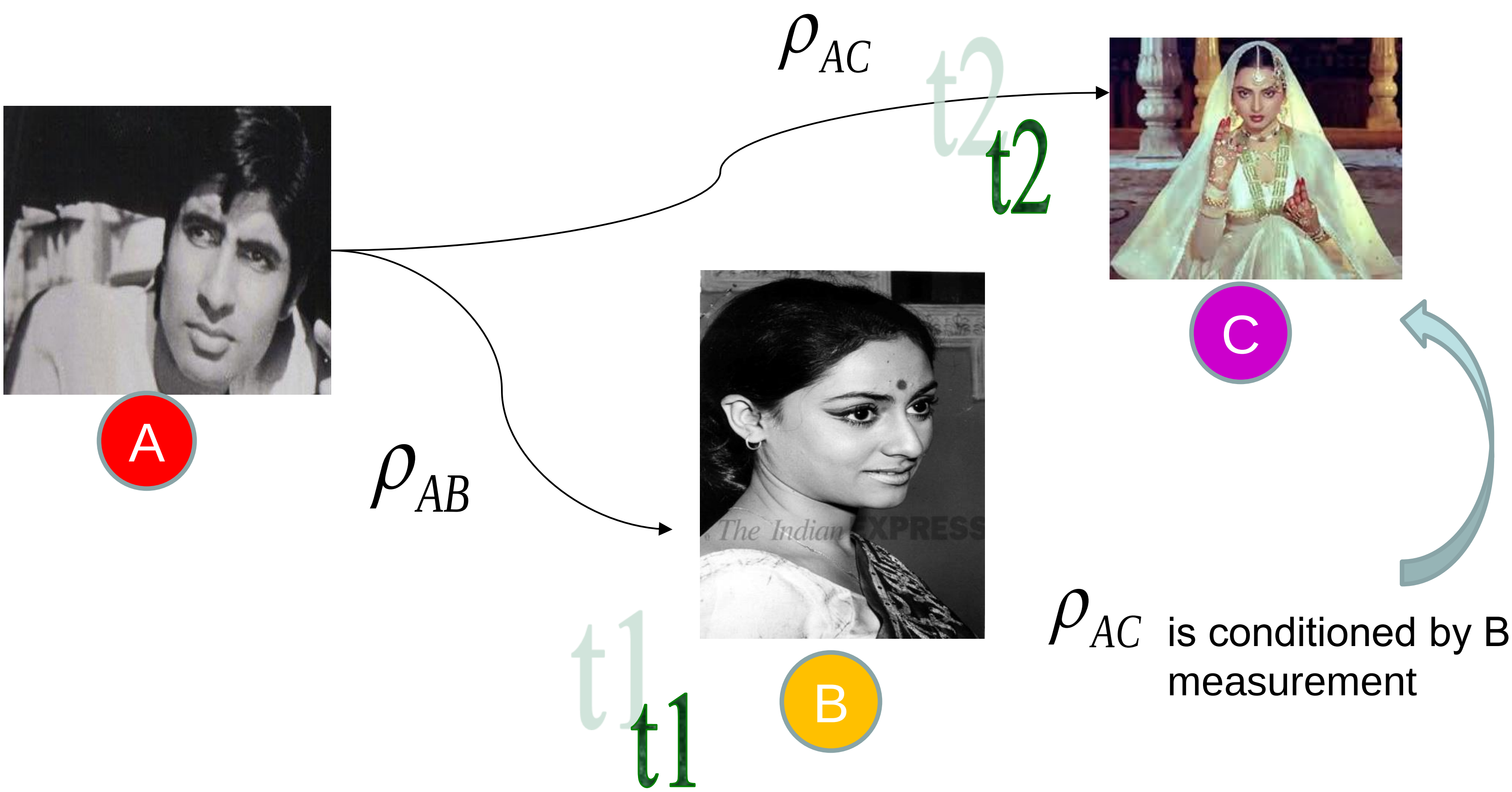


A can be maximally entangled with either B or C.
Otherwise, *no-signaling* is violated.

However, Alice can sequentially get entangled with Bob <u>and</u> Charlie

No-signaling is not applicable in this situation

Sequential entanglement



ρ_{AC} is conditioned by B's measurement

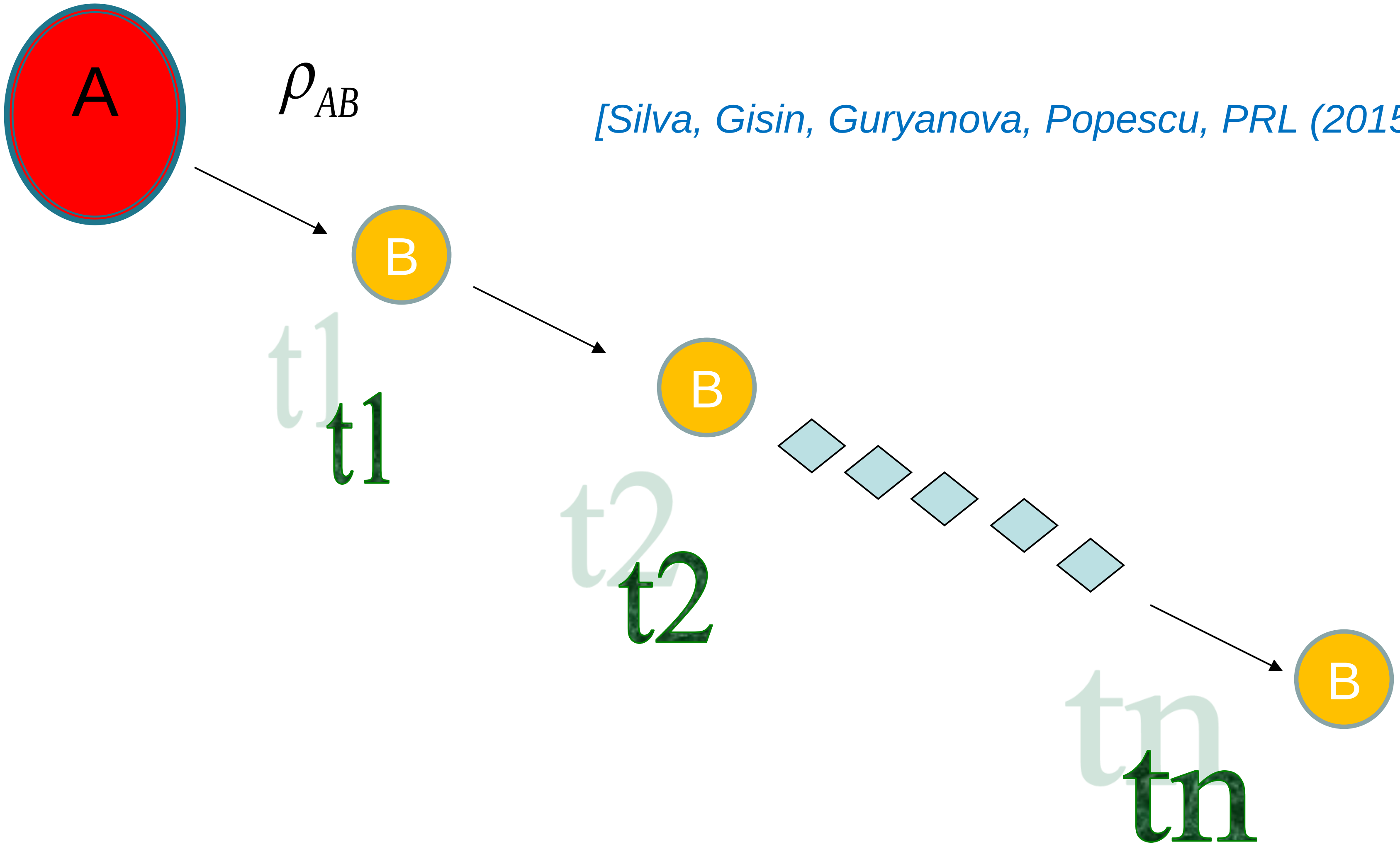
Sharing of nonlocality

- **Q:** How does one reveal nonlocality between Alice of one side and sequential Bobs on the other ?

- **A:** Check violation of CHSH inequality by the different pairs, e.g., Alice-Bob(1); Alice-Bob(2),….. Alice-Bob(n)

- **Note:** Bob(1), Bob(2)…..Bob(n-1) must perform POVMs; otherwise, no entangled state remains for the subsequent (n-th) Bob(n).

Sharing of nonlocality
(How many Bobs ?)

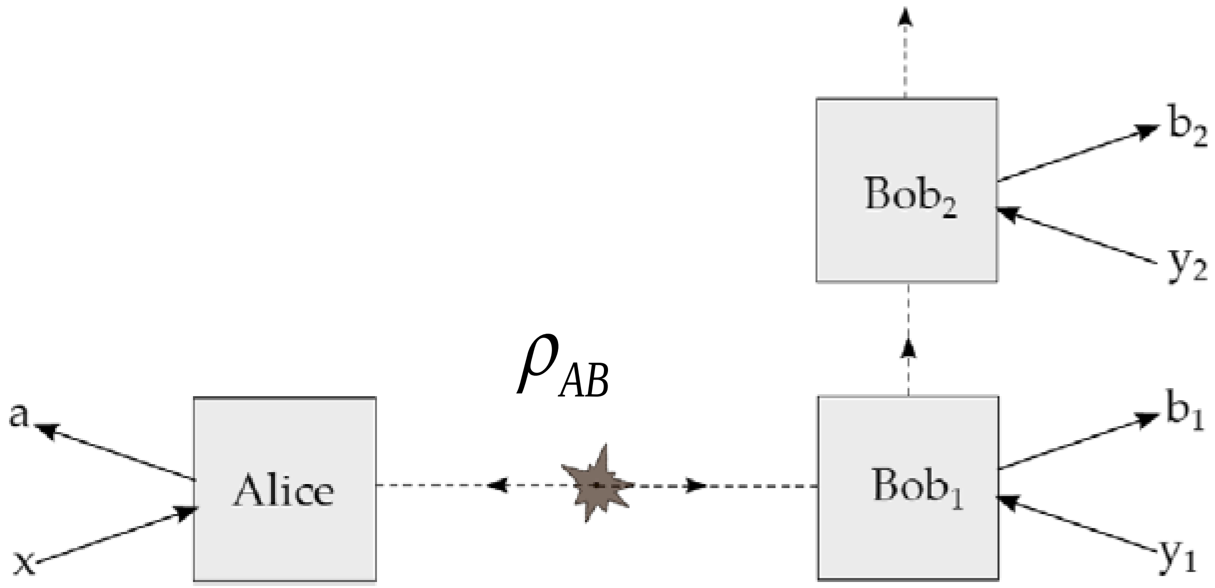


[Silva, Gisin, Guryanova, Popescu, PRL (2015)].

Definition of the problem

- Alice has access to one particle of a pair of entangled spin ½ particles.
- Series of Bobs (B(1),B(2),….B(n)) on the other side who can access the other particle sequentially but independently of each other (no-signalling condition NOT applicable).
- Alice performs a projective measurement (dichotomic input and output [0,1]).
- B(1),B(2)….B(n-1) perform one-parameter POVMs (dichotomic input and output [0,1]); B(n) may perform a projective measurement.
- Unbiased input settings for Alice and all Bobs, e.g., frequency of receiving input 0 and input 1 is same.

Bell scenario involving Alice and multiple Bobs

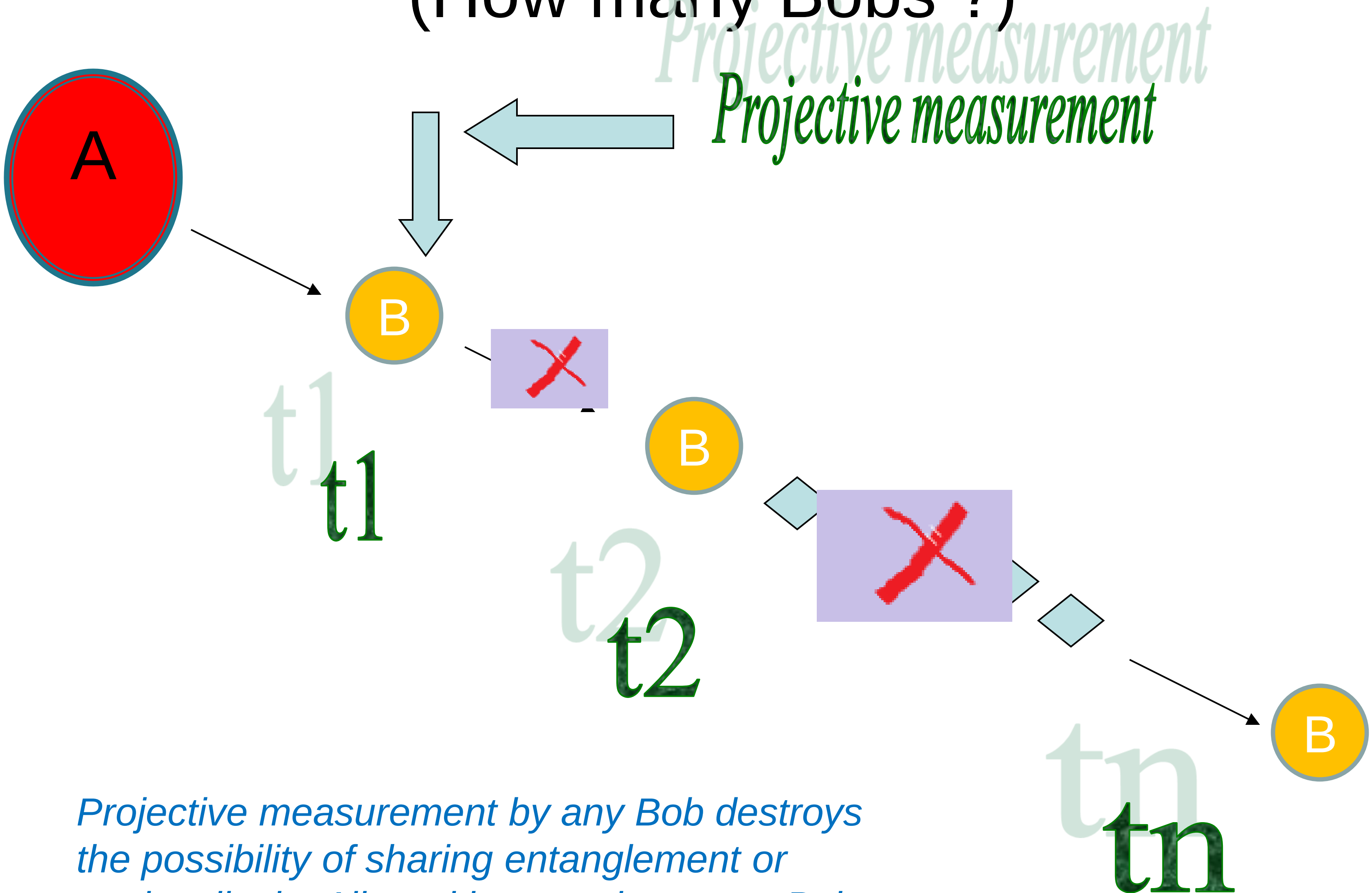


One spin-1/2 particle of an entangled pair is accessed by Alice. The other is accessed sequentially by the Bobs. No biasing of measurement inputs is allowed

Definition of the problem……

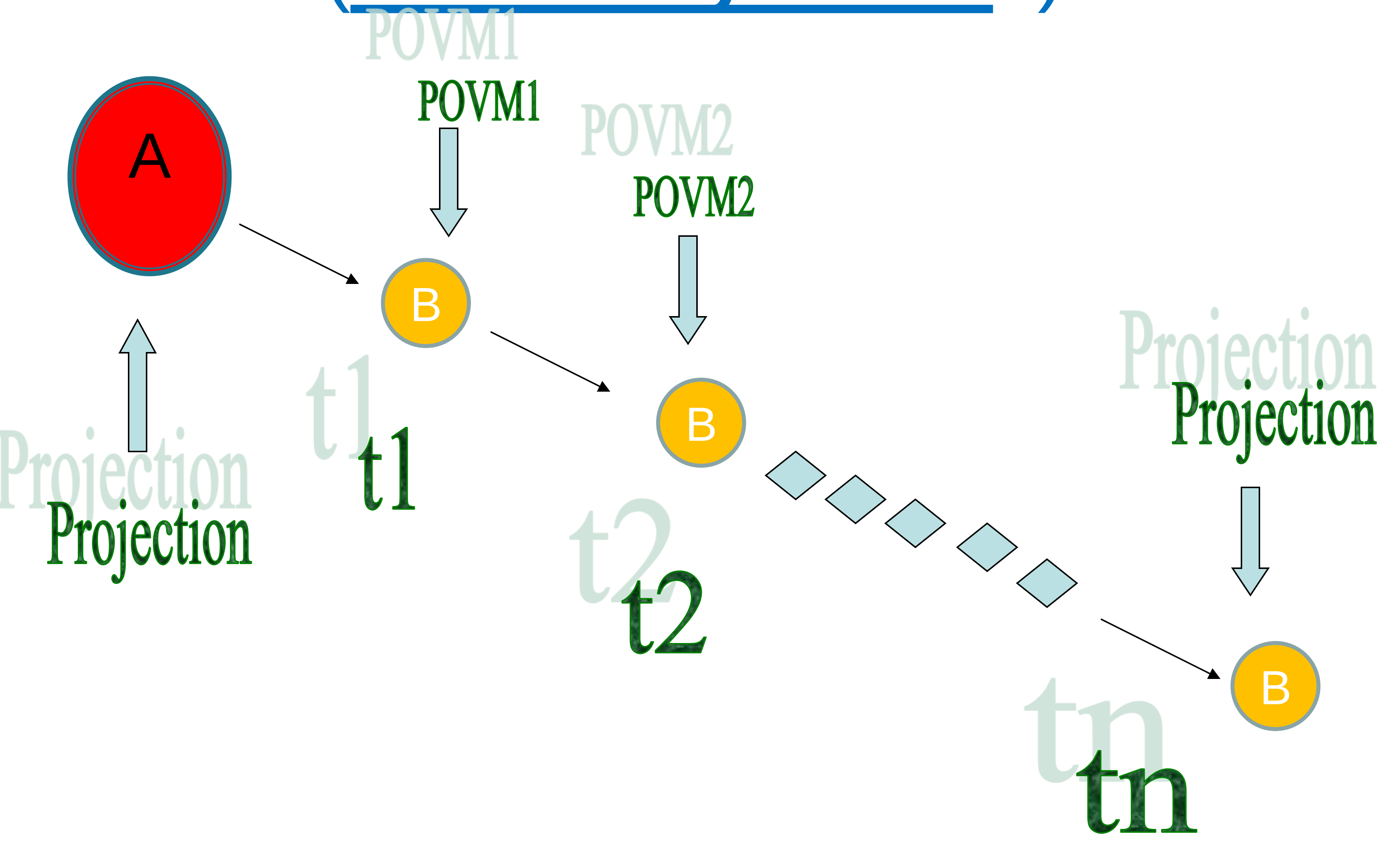
- To check for how many pairs the CHSH inequality is violated by (Alice-B1), (Alice-B2)…….(Alice-BN).
- Upper limit N=2 conjectured numerically for unbiased settings *[Silva, Gisin, Guryanova, Popescu, PRL (2015)].*
- We provide analytical proof using optimality of information gain versus disturbance trade-off with 1-parameter POVMs.
- Utility of the POVM formalism (or unsharp measurement)

Sharing of nonlocality
(How many Bobs ?)



Projective measurement by any Bob destroys the possibility of sharing entanglement or nonlocality by Alice with any subsequent Bob.

Sharing of nonlocality
(**How many Bobs** ?)



To determine the max. no. of Bobs we have to apply POVMs

POVMs *provide* ***optimality*** *in the* ***trade-off*** *between* ***information gain*** *and* ***disturbance*** *for an* ***unsharp measurement***

Information gain versus disturbance trade-off

System state

$$|\psi\rangle (= \alpha|0\rangle + \beta|1\rangle)$$

Apparatus state

$$\phi(q)$$

Joint system-apparatus state

$$\alpha|0\rangle \otimes \phi(q-1) + \beta|1\rangle \otimes \phi(q+1)$$

Reduced system state:

$$\rho' = F\rho + (1-F)(\pi^+\rho\pi^+ + \pi^-\rho\pi^-)$$

Quality factor:

$$F(\phi) = \int_{-\infty}^{\infty} \langle \phi(q+1)|\phi(q-1)\rangle dq$$

Probability of outcomes:

$$p(\pm) = G\langle\psi|\pi^\pm|\psi\rangle + (1-G)\frac{1}{2}$$

Precision of measurement:

$$G = \int_{-1}^{1} \phi^2(q)dq$$

G=1: Sharp measurement

<u>Information gain versus disturbance</u>

Optimality condition: best trade-off
(largest precision G for a given quality factor F)

$$F^2 + G^2 = 1$$

[Obtained by Silva et al. numerically using various pointer states]

One-parameter POVM
(Unsharp measurement with effect parameter):

$$E^{\lambda}_{\pm} = \lambda P_{\pm} + \frac{1-\lambda}{2}\mathbb{I}$$

(*System after pre-measurement)*
Luder transformation

$$\rho' = \sqrt{1-\lambda^2}\rho + (1-\sqrt{1-\lambda^2})(P_+\rho P_+ + P_-\rho P_-)$$

Probabilities for outcomes

$$p(\pm) = tr[E^{\lambda}_{\pm}\rho] = \lambda tr[P_{\pm}\rho] + \frac{1-\lambda}{2}$$

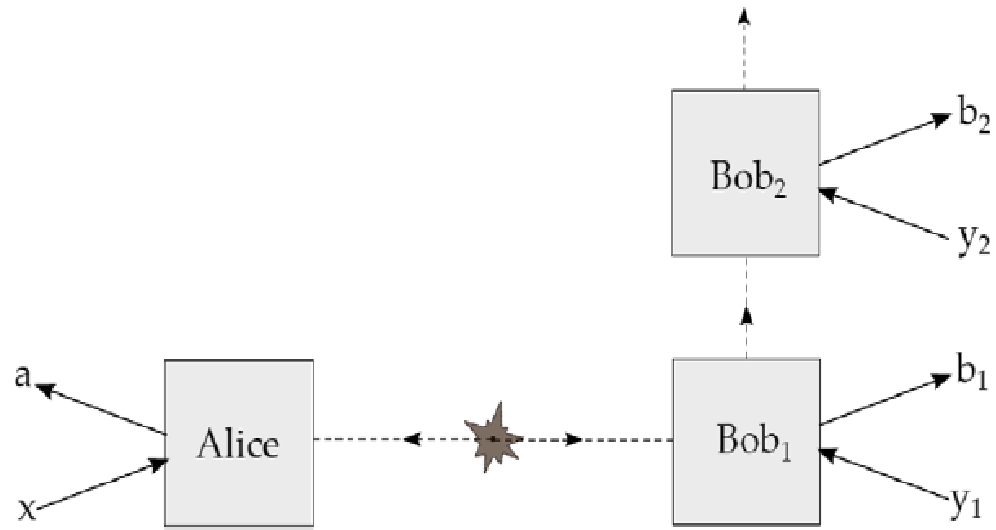
Relation between G, F and λ $\quad G = \lambda \quad F = \sqrt{1-\lambda^2}$

Limit of sharp measurement: $\quad G = \lambda = 1 \quad$ *F = 0*

Sharing of nonlocality

Two-qubit maximally entangled state

Bell-CHSH scenario



$$\frac{1}{\sqrt{2}}\left(|10> + |01>\right)$$

Alice's setting $\hat{X}$ and $\hat{Z}$

Bobs' setting $\frac{-(\hat{Z}+\hat{X})}{\sqrt{2}}, \frac{-\hat{Z}+\hat{X}}{\sqrt{2}}$ ***(Achieves Tsirelson's bound)***

(orthogonal measurements): ***1-parameter POVMs by Bobs*** *(n-1)*

No bias: Inputs $x, y_1, y_2 \in [0,1]$ with equal frequency

Sharing of nonlocality

Joint probability of getting outcome `*a*' by Alice and b_n by n-th Bob:

$$p(a,b_n) = p(a)p(b_n|a) = \frac{1}{2}Tr[\frac{\mathbb{I}+\lambda_n b_n \hat{y_n}.\vec{\sigma}}{2}\rho_{n|y_1...y_{n-1}}]$$

Case: Two Bobs

Joint probability:

$$p(a,b_2) = \frac{\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{y_2}.\hat{x}}{2} + \frac{1-\sqrt{1-\lambda_1^2}}{2}\frac{1-ab_2\lambda_2\hat{x}.\hat{y_1}\hat{y_1}.\hat{y_2}}{2}$$

First Bob measures weakly:

$$CHSH_{AB_1} = 2\sqrt{2}\lambda_1$$

Second Bob measures sharply:

$$CHSH_{AB_2} = \sqrt{2}(1+\sqrt{1-\lambda_1^2})$$

Precision range for violation by both Bobs:

$$\lambda_1 \in \left[1/\sqrt{2}, \sqrt{2\left(\sqrt{2}-1\right)}\right]$$

Sharing of nonlocality

Three Bobs

1st and 2nd Bob measure weakly; 3rd Bob measures sharply

Joint probability (CHSH correlation between Alice and Bob-3)

$$C_3 = \lambda_3[\sqrt{1-\lambda_1^2}\sqrt{1-\lambda_2^2}\hat{y}_3.\hat{x} + (1-\sqrt{1-\lambda_1^2})\sqrt{1-\lambda_2^2}\hat{x}.\hat{y}_1\hat{y}_1.\hat{y}_3$$
$$+ \sqrt{1-\lambda_1^2}(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y}_2\hat{y}_2.\hat{y}_3 + (1-\sqrt{1-\lambda_1^2})(1-\sqrt{1-\lambda_2^2})\hat{x}.\hat{y}_1\hat{y}_1.\hat{y}_2\hat{y}_2.\hat{y}_3]$$

Bob is ignorant about inputs of previous Bobs
(average over all possible earlier inputs)
Averaged correlation between Alice and Bob-3

$$\bar{C}_3 = \sum_{y_1 y_2} C_3 P(y_1) P(y_2)$$

Proof of impossibility of sharing nonlocality with more than two Bobs

Averaged CHSH correlation between Alice and Bob-3

$$\mathcal{I}^3 = \frac{(1+\sqrt{1-\lambda_1^2})(1+\sqrt{1-\lambda_2^2})}{\sqrt{2}}$$

CHSH A-B1: $2\sqrt{2}\lambda_1$

CHSH A-B2: $\lambda_2\sqrt{2}(1+\sqrt{1-\lambda^2})$

Range of sharpness for violation: If $\lambda_1 > 1/\sqrt{2}$ and $\lambda_2 > \frac{2}{\sqrt{2}+1}$

$$CHSH_{AB_3} \leq 2$$

$$CHSH_{AB_i} = 1.88$$

Violation not possible by all three Bobs. Max. Violation

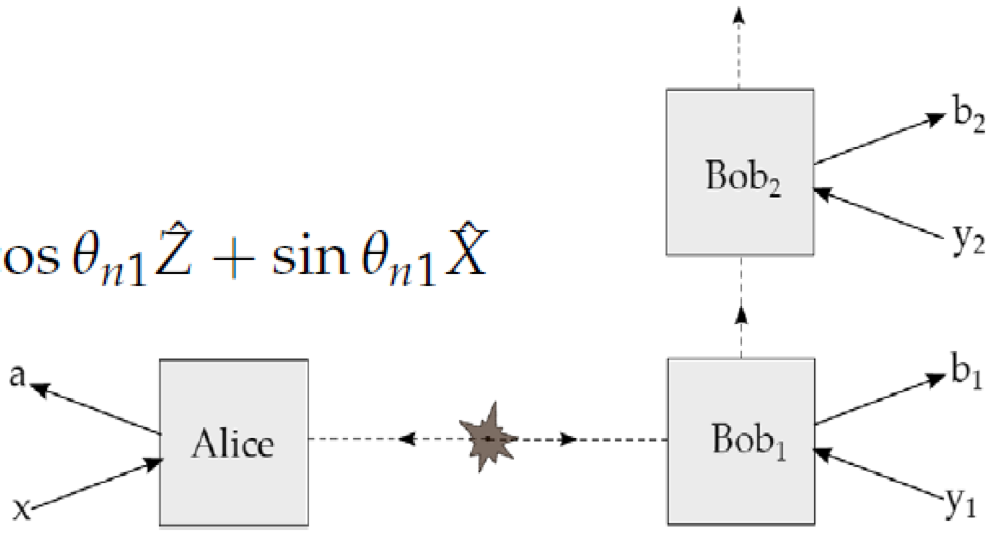
If other two Bobs obtain max violation *(i=1,2,3)*

Non-orthogonal measurements: Is there any advantage ?

Alice's settings: $\hat{X}$ and $\hat{Z}$

Bob's settings:

$$y_n^0 = \cos\theta_{n0}\hat{Z} + \sin\theta_{n0}\hat{X}, y_n^1 = \cos\theta_{n1}\hat{Z} + \sin\theta_{n1}\hat{X}$$



Correlations (Alice-Bob1):

$$\mathcal{I}^1(xy_1) = \lambda_1\tilde{\mathcal{I}}^1(xy_1)$$

$$\tilde{\mathcal{I}}^1(xy_1) = (\cos[\theta_{10}] - \cos[\theta_{11}] + \sin[\theta_{10}] + \sin[\theta_{11}])$$

Correlations for non-orthogonal measurement settings:

Alice-Bob2:

$$\mathcal{I}^2(x(y_1)y_2) = \lambda_2[(1-F_1)\tilde{\mathcal{I}}^2(x(y_1)y_2) + F_1\tilde{\mathcal{I}}^1(xy_2)]$$

$$\tilde{\mathcal{I}}^2(x(y_1)y_2) = \frac{1}{2}\sum_{i,j=0}^{1}((-1)^j\cos\theta_{1i} + \sin\theta_{1i})\cos(\theta_{1i} - \theta_{2j})$$

Alice-Bob3:

$$\mathcal{I}^3(x(y_1y_2)y_3) = \frac{F_1+F_2}{2}\tilde{\mathcal{I}}^1(xy_3) + \frac{(1-F_1)F_2}{2}(\tilde{\mathcal{I}}^2(x(y_1)y_3) - \tilde{\mathcal{I}}^2(x(y_1+\frac{\pi}{2})y_3))$$
$$+ \frac{(1-F_2)F_1}{2}(\tilde{\mathcal{I}}^2(x(y_2)y_3) - \tilde{\mathcal{I}}^2(x(y_2+\frac{\pi}{2})y_3)) + \frac{(1-F_1)(1-F_2)}{16}\tilde{\mathcal{I}}^3(x(y_1y_2)y_3)$$

$$F_i = \sqrt{1-\lambda_i^2}$$

No improvement in violation:

Example:

$$y_1^0 \approx 0.19\hat{Z} + 0.98\hat{X} \qquad y_1^1 \approx -0.19\hat{Z} + 0.98\hat{X}$$

$$y_2^0 \approx 0.19\hat{Z} + 0.98\hat{X} \qquad y_2^1 \approx -0.19\hat{Z} + 0.98\hat{X}$$

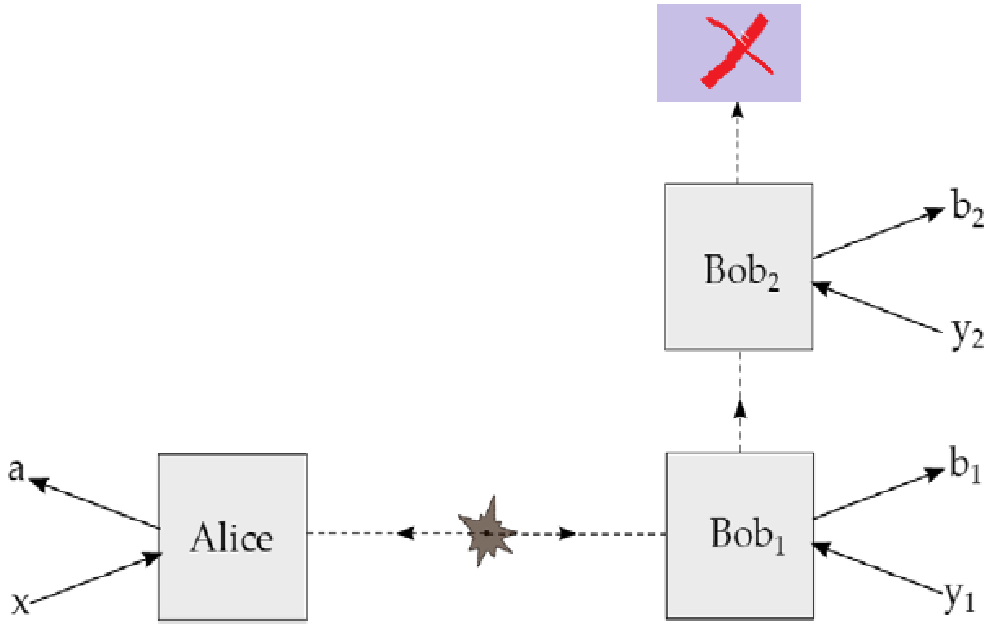
$$y_3^0 \approx 0.04\hat{Z} + 0.99\hat{X} \qquad y_3^1 \approx -0.04\hat{Z} + 0.99\hat{X}$$

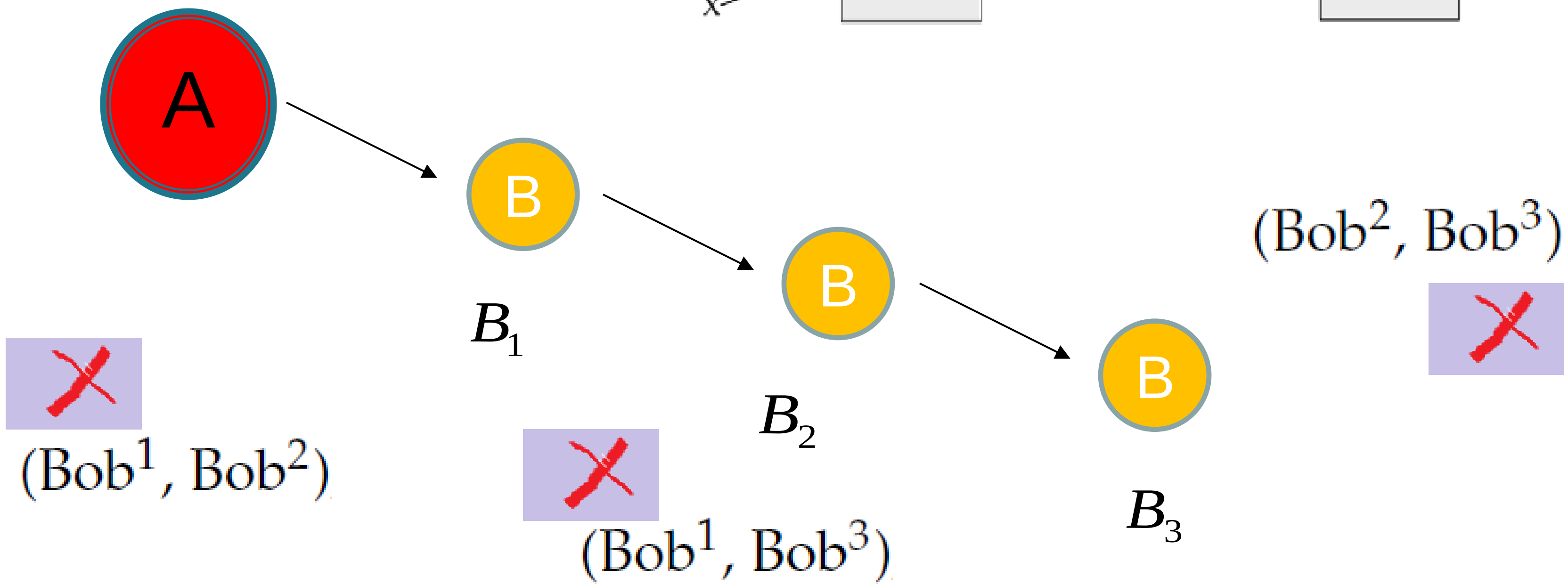
$$I_1 = 2.1 \qquad I_2 = 2.1 \qquad I_3 \rightarrow 1.89$$

No CHSH violation possible with more than 3 Bobs

Sequence NOT important

Alice can obtain violations with any two pairs





Sharing of steerability

Steering: Weaker form of nonlocality than Bell-violation

(All Bell-violating states are strict subset of all steerable states)

How many Bobs acting sequentially and independently of each other can steer Alice's state ?

Necessary and sufficient steering condition in 2-2-2 scenario

(two-qubit shared state; two parties, two measurement settings per party)

[Cavalcanti, Foster, Fura, Wiseman, (2015)]

CFFW Inequality

$$S_{BA} = \sqrt{\langle (B+B')A\rangle^2 + \langle (B+B')A'\rangle^2} + \sqrt{\langle (B-B')A\rangle^2 + \langle (B-B')A'\rangle^2} \leq 2.$$

Violation of CFFW inequality by Alice and two Bobs

Alice's settings: $\{\hat{x^0}, \hat{x^1}\}$

Bob's settings: $\{\hat{y_n^0}, \hat{y_n^1}\}$

Correlation between Alice and Bob1 : $C_1^{ji} = -\lambda_1 (\hat{y_1^i}.\hat{x^j})$

Average correlation between Alice and Bob2: $\overline{C_2^{jk}} = \sum_{i=0,1} C_2^{jk} P(y_1^i)$

Necessary and sufficient steering condition for 3rd Bob:

$$S_n = \sqrt{(\overline{C_n^{00}} + \overline{C_n^{01}})^2 + (\overline{C_n^{10}} + \overline{C_n^{11}})^2} + \sqrt{(\overline{C_n^{00}} - \overline{C_n^{01}})^2 + (\overline{C_n^{10}} - \overline{C_n^{11}})^2}$$

Violation not possible by more than two Bobs

Violation of 3-settings inequality by Alice and three Bobs

n-settings inequality:
[Cavalcanti-Jones-Wiseman-Reid (2009)]

$$F^n = \frac{1}{\sqrt{n}} \Big| \sum_{i=1}^{n} \langle A_i \otimes B_i \rangle \Big| \leq 1$$

Compute average correlation functions between Alice and i-th Bob:

$$\overline{C_n^{jk}}$$

CJWR function between Alice and n-th Bob for 3-settings:

$$F_n^3 = \frac{1}{\sqrt{3}} \Big| \sum_{i=1}^{3} \overline{C_n^{ii}} \Big|$$

Bob1, Bob2 and Bob3 can steer Alice !

<u>Sharing of nonlocality for two-qubit state: Summary</u> *[S. Mal, A. S. Majumdar, D. Home, Mathematics <u>4</u>, 48 (2016); arXiv; …. (2017)]*

- With how many sequential Bobs can Alice obtain CHSH violation ?
  [Silva, Gisin, Guryanova, Popescu, PRL (2015)]

- Trade-off between information gain and disturbance in a measurement is optimized by one-parameter POVM.

- **Alice (measuring sharply) on one side cannot obtain CHSH violation with more than two Bobs on other side.** ***(****Result valid for unbiased measurement settings only).*

- Non-orthogonal measurement settings provide no advantage.

- Steerabilty between Alice and three Bobs using CJWR inequality.