Quantum coherence and Quantum Discord

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Plan of the Talk

Introduction

- Quantum superposition is the characteristic feature of quantum mechanics. Evolution of a multi-partite quantum system can not be efficiently captured on a probabilistic classical computers because of superposition (R. P. Feynman 82).
- Although a density matrix having only diagonal entries (incoherent states), evolving under operations that takes a diagonal matrix to another diagonal matrix are analogous to the classical evolution of a probability distribution through conditional probabilities.
- Based on above considerations, Aberg et al (arxiv:quant-ph/ 0612146) and Baumgratz et al (PRL14) have proposed resource theory of quantum superposition.
- Following the proposal of Aberg and Baumgratz, resource theory of coherence has been developed analogous to the resource theory of entanglement (Winter and Yang PRL016).

Economy of Quantum Coherence (Baumgratz et al PRL14)

- **Free state or Incoherent states**: These are the density states having no off-diagonal elements with respect to a preferred measurement basis $|i\rangle_{i=1,2,...d}$ (computational basis) in a Hilbert space of dimension d, i.e.

$$\rho_I = \sum_i \lambda_i |i\rangle \langle i| .$$

  with $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.

- **Free Operations**: A quantum operation is incoherent if there exists a Kruas representation with set of Kraus operators $\{\hat{K}_l\}$ if each Kraus operator maps an incoherent state to another incoherent state, i.e.

$$\hat{k}_l I \hat{K}_l \in I$$

  Here, I is the set of all incoherent states.

Incoherent Operations:Operations that do not generate quantum coherence

- Incoherent operations are fundamental to theory of decoherence for example: Bit flip channel $\rho' = p\sigma_x\rho\sigma_x + (1-p)\rho$ or phase flip channel $\rho\rho' = p\sigma_z\rho\sigma z + (1-p)\rho$.
- Any incoherent Kraus operator can be written as, $\hat{K}_\mu = \sum_i c(i) \left| f_\mu(i) \right\rangle \langle i|$, here, $\left| f_\mu(i) \right\rangle$ is again an element of the incoherent basis set $\{|i\rangle\}_{i=1,\ldots d}$.

Operations that do not utilize coherence (Yadin et al PRX15)

- Although operations $\hat{K}_\mu = \sum_i c(i) \left|f_\mu(i)\right\rangle \langle i|$ cannot generate coherence, they can utilize coherence. For example, quantum states $\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$ having same diagonal part in computational basis $\{|0\rangle, |1\rangle\}$ can be distinguished by incoherent Kraus operators, $\{\hat{K}_1 = \frac{|0\rangle\langle 0|+|0\rangle\langle 1|}{\sqrt{2}}, \hat{K}_2 = \frac{|1\rangle\langle 0|-|1\rangle\langle 1|}{\sqrt{2}}\}$
- A Quantum operation that do not use coherence operates only on the diagonal elements of a density matrix, i.e., for every Kraus operator $\hat{K}_l$, relation $tr(\hat{K}_l{}^\dagger \hat{K}^l \rho) = tr(\hat{K}_l{}^\dagger \hat{K}^l \rho^d)$ is satisfied.

Operations that do not utilize coherence

- Hence, before applying an incoherent operation, one can apply decohering operation $\Lambda(\rho) = \sum_i \langle i| \rho |i\rangle |i\rangle \langle i|$ and results will be same.
- These operations are the subset of incoherent operations and only Kraus operator $\hat{K_\mu} = \sum_i c(i) \left|f_\mu(i)\right\rangle \langle i|$ with invertible functions (one to one) $f_\mu(i)$ are allowed under this class of operations also known as *strictly incoherent operations*.
- A quantum process with incoherent operations only does not involve interference, hence can be simulated by classical probability distribution.

Coherence Measure

A coherence measure $C(\rho)$, for any density matrix $\rho \in H$, should satisfy following constraint:

- $C(\delta) = 0$ if and only if δ is an incoherent state.
- $C(\rho)$ must not increase under an incoherent operation;

$$C(\sum_i \hat{K}_i \rho \hat{K}_i^\dagger) \leq C(\rho).$$

Here, observer does not have access to individual Kraus operators $\hat{K}_l$ of the operator Δ.

Coherence Measure

- $C(\rho)$ must not increase even if the observer has access to individual measurement outcome; $\sum_i p_i C(\rho_i) \leq C(\rho)$, where $\rho_i = \hat{K}_i \rho \hat{K}_i^\dagger / p_i$, $p_i = tr(\hat{K}_i \rho \hat{K}_i^\dagger)$, with $\hat{K}_i$ being an incoherent operation.
- $C(\rho)$ must not increase under mixing of quantum states; $\sum_i p_i C(\rho_i) \leq C(\sum_i p_i \rho_i)$.

Coherence Measure

Following are two measures of coherence satisfying all four constraints;

- *Relative entropy Based Measure:* $C_r(\rho) = min_{\sigma \in I} S(\rho||\sigma) = S(\rho^d) - s(\rho)$, here, $S(\rho^d)$ is the diagonal part of density matrix ρ with respect to the reference basis, $\{|i\rangle\}$.
- l_1 *norm based measure:* $C_{l_1}(\rho) = \sum_{i,j,i \neq j} |\rho_{ij}|$

Relative Entropy Measure

- Relative entropy measure can be extended to multipartite settings for entropy based measures. A bipartite density matrix σ_{AB} will have no local coherence or said to be locally incoherent on Bob side with respect to a preferred basis set $\{|i\rangle_B\}$, if it can be written as,

$$\sigma_{AB} = \sum_k p_k \sigma_A^k \otimes \zeta_B^k.$$

here ζ_B^k is an incoherent state: $\zeta_B^k = \sum_i p_i |i\rangle \langle i|$.

- Coherence measure for system B can be given in similar fashion in the multipartite scenario: $C_r^{A|B}(\rho_{AB}) = min_{\sigma_{AB} \in QI} S(\rho_{AB}||\sigma_{AB})$ here, QI is the set of all locally incoherent states. It can also be written as,

$$C_r^{A|B}(\rho_{AB}) = S(\rho_{AB}^{d_B}) - S(\rho_{AB}),$$

- Quantum coherence in multipartite scenario has been probed by Yadin et al (PRX16), Yao et al (PRA15).

Quantum Coherence and Holevo Bound

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Let's consider a quantum instrument O which takes quantum state ρ as an input and outputs a quantum state from the set $\{\rho_i\}$ with corresponding probability $\{p_i\}$,

$$\rho \xrightarrow{\text{O},\ p_i} \sigma_i.$$

The information that an observer can gain from this apparatus after performing a POVM can never exceed the Holevo Bound, $\chi = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$; the Bound is achievable in asymptotic limit.

Quantum Coherence and Holevo Bound

- Let's consider a scenario where observer can only perform strictly incoherent operations (operations that do not use coherence). As mentioned above before performing a strictly incoherent operation one can apply decohering operation $\Delta[\rho] = \rho^d$ and it would not affect the measurement probabilites. Thus, maximum information that an observer can get from instrument O under strictly incoherent operation would never exceed $S(\sum_x p_x \rho_x^d) - \sum_x p_x S(\rho_x^d)$ and it's achievable by performing a measurement in the incoherent basis set, $\{|i\rangle\}$, i.e.,

$$H(X:Y) = S(\rho^d) - \sum_x p_x S(\rho_x^d)$$

Quantum Coherence and Holevo Bound

- If receiver Bob is only allowed to perform strictly incoherent operations, loss in the information gain from the instrument O is equal to the loss of coherence due to mixing.

$$IL = \chi - H(X:Y) \implies S(\rho) - \sum_x p_x S(\rho_x) - S(\rho^d) - \sum_x p_x S(\rho_x^d), \tag{1}$$

$$CL = \sum_x p_x C_r(\rho_x) - C_r(\sum_x p_x \rho_x). \tag{2}$$

From 1 and 2, we have, IL= CL.

- If quantum instrument O is coherence erasing, i.e. $C(\sum_i p_i \rho_i) = 0$, then loss of information under strictly incoherent operation is equivalent to the average coherence of the ensemble $\{\rho_i\}$, which is $\sum_i p_i C(\rho^d)$.

Quantum Discord

Henderson and Vedral JOPA(2001), Oliver and Zurek PRL2002,

- Let us consider a bipartite density matrix ρ_{AB}, the quantity $\delta(A:B)$ has been defined as the difference of quantum mutual information $I(A:B)$ and mutual information $J(A:B)$ after performing a measurement on system B,

$$\delta(A:B) = I(A:B) - J(A:B)$$

$$I(A:B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$$

and

$$J(A:B) = S(\rho^B) - S(\rho_{AB}|\{\Pi_i^B\})$$

or equivalently,

$$\delta(A:B) = S(\rho_B) - S(\rho_{AB}) + S(\rho_{AB}|\{\Pi_i^B\}).$$

Quantum Coherence and Quantum Discord

- Let Bob choose a projector $|i\rangle_B \langle i|$ to perform measurement; $S(\rho_{AB}|\{\Pi_i^B\}) = \sum_i p_i S(\rho_i)$. Here, $p_i = tr_{AB}(\rho_{AB} |i\rangle_B \langle i|)$ and $\rho_i = tr_B(\rho_{AB} |i\rangle_B \langle i|) \otimes |i\rangle \langle i| / p_i$.
- Quantum discord $D(A:B)$ is the minimum value of $\delta(A:B)$ over all the projectors $\{\Pi_j^B\}$.
  Using the expression,

$$S(\sum_i p_i \rho_i) = S(\rho_{AB}^d) = H(p_i) + \sum p_i S(\rho_i),$$

one obtains,

$$\delta(A:B) = S(\rho_B) - S(\rho_{AB}) + S(\rho_{AB}^d) - H(p_i).$$

Quantum Coherence and Quantum Discord

- $\delta(A:B)$ can be written in terms of coherence measures,

$$\delta(A:B) = C_r^{(A|B)}(\rho^{AB}) - C_{rel.ent}(\rho_B),$$

Here, $C_r^{(A|B)}(\rho^{AB})$ is coherence of subsystem B in multipartite scenario as mentioned earlier.

- We also have,

$$D(A:B) = min_{|i\rangle\langle i|}(C_r^{(A|B)} - C_{rel.ent}(\rho_B)).$$

- Using convexity of coherence measures, it's easy to show that for a separable quantum state $\rho^{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B$,

$$\delta(A:B) \leq \sum_i p_i C_r(\rho_{ib}) - C_r(\sum_i p_i \rho_{ib}), \quad (3)$$

- If we throw qubit A into a dustbin which is quantum correlated with the system B, it will decrease the coherence of subsystem B. This can be illustrated by following example:
- Classically Correlated system: $\sum_{ij} p_{ij} |i\rangle^A \langle i| \otimes |j\rangle^B \langle j|$, average coherence of system B is zero. Thus tracing out subsystem A will not have any effect on coherence of subsystem B.
- Separable quantum system: $\rho^{AB} = \sum_i p_i |i\rangle^A \langle i| \otimes \rho_i^B$, Average coherence of subsystem B is $\sum_i p_i C(\rho_i^B)$. After Tracing out subsystem A, the coherence of subsystem B, is $C(\sum_i p_i \rho_i^B) \leq \sum_i p_i C(\rho_i^B)$.

- Let take a look again at following relation,

$$\delta(A:B) \leq \sum_i p_i C_r(\rho_{ib}) - C_r(\sum_i p_i \rho_{ib}), \tag{4}$$

- R.H.S is the loss of coherence due to mixing, which is equivalent to $IL = \chi - H(X:Y)$ as mentioned earlier. Hence, we have,

$$\delta(A:B) \leq IL_b \tag{5}$$

- Let's consider a basis Y_{max} for which the accessible information about the ensemble $\{\rho_i^B\}$ is maximum. We have the following relation for this basis:

$$\delta(A:B)_{Y_{max}} \leq IL_{b_{min}} \tag{6}$$

- It is is easy to see that quantum discord, which is minimum of $\delta(A:B)$ over all possible projective measurements on subsystem B is also bounded by $IL_{b_{min}}$.

$$D(A:B) \leq IL_{b_{min}} \tag{7}$$

- Using relation, $IL_{b_{min}} = \chi - H(X:Y_{max})$ we have complementary relation between quantum discord and maximum accessible information $H(X:Y_{max})$ of Bob's ensemble.

$$D(A:B) + H(X:Y_{max}) \leq \chi \tag{8}$$

- It is obvious from equation 8 that if Holevo bound is achievable by performing a measurement on Bobs system, there will be no quantum discord; D(A : B) = 0.

Conclusion

- For a classical quantum state $\sum_i p_i \rho_i^A \otimes |i\rangle^B \langle i|$, Quantum discord $D(A:B)$ is zero because the ensemble on Bob's side is completely distinguishable.
- We show that for separable states $\sum_i p_i \rho_i^A \otimes \rho_i^B$, Quantum Discord $D(A \leftarrow B)$ is zero if there exists a measurement basis which achieves Holevo Bound on the ensemble $\{\rho_i^B\}$.