Aspects Of NonMarkovianity In Quantum Walks

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Motivation

- After motivating the need for a study of Open Quantum Systems, we introduce, briefly, quantum operations and use it to discuss a few well known noise processes.
- We then move to some recent developments in the efforts to understand non-Markovian phenomenon.
- Our discussion about non-Markovian behaviour would be made in the backdrop of some well-known non-Markovian processes.
- This will be done using the platform of Quantum Walks.

- The theory of open quantum systems addresses the problems of damping and dephasing in quantum systems by the assertion that all real systems of interest are 'open' systems, surrounded by their environments [U. Weiss: (1999); H. -P. Breuer and F. Petruccione: (2002)].
- Quantum optics provided one of the first testing grounds for the application of the formalism of open quantum systems [W. H. Louisell: (1973)], [G. S. Agarwal: (1973)]. Application to other areas was intensified by the works of [Caldeira and Leggett: (1983)], [Grabert, Schramm and Ingold: 1988)] and [Zurek: (1993)], among others.
- The recent upsurge of interest in the problem of open quantum systems is because of the spectacular progress in manipulation of quantum states of matter, encoding, transmission and processing of quantum information, for all of which understanding and control of the environmental impact are essential [Turchette *et al.*: (2000); Myatt *et al.*: (2000); Haroche *et al.* (1996)]. This increases the relevance of open system ideas to quantum computation and quantum information.

- Hamiltonian of the total (closed system):

$$H = H_S + H_R + H_{SR}.$$

- S- system, R- reservoir (bath), $S - R$-interaction between them.
- System-reservoir complex evolves unitarily by:

$$\rho(t) = e^{-\frac{i}{\hbar}Ht}\rho(0)e^{\frac{i}{\hbar}Ht}.$$

- We are interested in the reduced dynamics of the system S, taking into account the influence of its environment. This is done by taking a trace over the reservoir degrees of freedom, making the reduced dynamics *non-unitary*:

$$\rho^s(t) = \mathrm{Tr}_{\mathrm{R}}(\rho(\mathrm{t})) = \mathrm{Tr}_{\mathrm{R}}\left[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\mathrm{Ht}}\rho(0)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathrm{Ht}}\right].$$

- Open quantum systems can be broadly classified into two categories:
  (A). Quantum non-demolition (QND), where $[H_S, H_{SR}] = 0$ resulting in decoherence without any dissipation [Braginsky *et al.*: (1975), (1980); Caves *et al.*: (1980); G. Gangopadhyay, S. M. Kumar and S. Duttagupta: (2001); SB and R. Ghosh: (2007)] and
  (B). Quantum dissipative systems, where $[H_S, H_{SR}] \neq 0$ resulting in decoherence with dissipation [Caldeira and Leggett: (1983); H. Grabert, P. Schramm and G-L. Ingold: (1988); SB and R. Ghosh: (2003), (2007)].
- In the parlance of quantum information theory, the noise generated by a QND open system would be a "phase damping channel", while that generated by a dissipative (Lindblad) evolution would be a "(generalized) amplitude damping channel".

- The open system evolution starts from the system and the reservoir being initially separated or correlated.
  (A). Separable Initial Condition: it is assumed that the system and the environment (reservoir) are initially uncorrelated [Feynman and Vernon: (1963); Caldeira and Leggett: (1983)]. In such a situation the initial density matrix factorizes so that

$$\rho(0) = \rho_S(0).\rho_R(0),$$

  where $\rho_S(0)$ stands for the initial system density matrix and $\rho_R(0)$ stands for the initial reservoir density matrix.

(B). Non-Separable Initial Condition: in many applications, the system and the reservoir are integral parts of the same system and their interaction is not at our disposal. These considerations lead to the introduction of a class of initial conditions, the 'generalized initial conditions' [Hakim and Ambegaokar: (1985); H. Grabert, P. Schramm and G-L. Ingold: (1988); SB and R. Ghosh: (2003)]. A very general class of initial conditions are of the form

$$\rho_0 = \sum_j O_j \rho_\beta O_j',$$

where

$$\rho_\beta = Z_\beta^{-1} \exp(-\beta H)$$

is the canonical density matrix describing the equilibrium of the interacting system in the presence of a time-independent potential V and Z_β^{-1} is the partition function. Here $\beta = (k_B T)^{-1}$, with T being the equilibrium temperature of the interacting system. The operators O_j, O_j' act upon the system coordinate only and leave the environment (reservoir) coordinates unchanged but can be chosen arbitrarily otherwise.

Time scales associated with the Open System Evolution

- The open system evolution is characterized by a number of time-scales, the salient ones being:
- Scale associated with the natural frequency of the system.
- Relaxation time scale determined by the S-R coupling strength.
- Reservoir correlation time (memory time) associated with the high-frequency cutoff in the reservoir spectral density and the time scale associated with the reservoir temperature, which measures the relative importance of quantum to thermal effects.

- Any evolution consistent with the general rules of quantum mechanics can be described by a linear, completely positive map, called quantum operation ($\mathcal{E}$). [M. A. Nielsen and I. L. Chuang: (2000)]
- Complete positivity: Consider any positive map $\mathcal{E}$ on the system Q_1: if an extra system R of arbitrary dimensionality is introduced, and $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive on any positive operator A on the combined system RQ_1, where $\mathcal{I}$ denotes the identity map on system R: then $\mathcal{E}$ is completely positive.
- A unitary evolution is a special case of a quantum operation: general quantum operations can describe non-unitary evolutions, due to coupling with environment.

- Any such quantum operation can be composed from elementary operations:
  - unitary transformations: $\mathcal{E}_1(\rho) = U\rho U^\dagger$
  - addition of an auxiliary system: $\mathcal{E}_2(\rho) = \rho \otimes \sigma$: here ρ is the original system and σ is the auxiliary one
  - partial traces: $\mathcal{E}_3(\rho) = \mathrm{Tr_B}(\rho)$
  - projective measurements: $\mathcal{E}_4(\rho) = P_k \rho P_k / \mathrm{Tr(P_k \rho)}$, with $P_k^2 = P_k$.

Connection to quantum noise processes

- Interpret our results in terms of familiar noisy channels. How these environmental effects can affect quantum computing.
  In operator-sum representation, action of superoperator $\mathcal{E}$ due to environmental interaction

  $$\rho \longrightarrow \mathcal{E}(\rho) = \sum_k \langle e_k|U(\rho \otimes |f_0\rangle\langle f_0|)U^\dagger|e_k\rangle = \sum_j E_j \rho E_j^\dagger,$$

  unitary U acts jointly on system-environment $|f_0\rangle$: environment's initial state; $\{|e_k\rangle\}$ a basis for the environment.
- environment-system assumed to start in a separable state.
- $E_j \equiv \langle e_k|U|f_0\rangle$ are the Kraus operators; partition of unity: $\sum_j E_j^\dagger E_j = \mathcal{I}$. Any transformation representatable as operator-sum is a completely positive (CP) map.

Connection to quantum noise processes: QND interactions

- Here we give some illustrations of single qubit quantum noisy channels.
- QND interactions yield quantum phase damping channel: uniquely non-classical quantum mechanical noise process, describing the loss of quantum information without the loss of energy.
- Kraus operator elements [SB and R. Ghosh: (2007)]

$$E_0 \equiv \begin{bmatrix} 1 & 0 \\ 0 & e^{i\beta(t)}\sqrt{1-\lambda} \end{bmatrix}; \qquad E_1 \equiv \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix},$$

  where $\beta(t)$ encodes the free evolution of the system and λ the effect of the environment.
- Applying this to initial state yields

$$\rho^s(t) = \mathcal{E}(\rho^s(0)) = \begin{pmatrix} \cos^2\left(\frac{\theta_0}{2}\right) & \left(\frac{1}{2}\right) e^{-i(\beta(t)+\phi_0)} \sin(\theta_0)\sqrt{1-\lambda} \\ \left(\frac{1}{2}\right) e^{i(\beta(t)+\phi_0)} \sin(\theta_0)\sqrt{1-\lambda} & \sin^2\left(\frac{\theta_0}{2}\right) \end{pmatrix}.$$

Connection to quantum noise processes: QND interactions

- Comparing with QND interaction with bath of harmonic oscillators

$$\lambda(t) = 1 - \exp\left[-2(\hbar\omega)^2\gamma(t)\right]; \qquad \beta(t) = \omega t.$$

- QND interaction with a bath of two level systems

$$\lambda(t) = 1 - (1 + 4\omega_c^2 t^2)^{(-\gamma_0/2\pi)(\hbar\omega)^2}; \qquad \beta(t) = \omega t.$$

- $\lambda(t) \longrightarrow 1$ as $t \longrightarrow \infty$ (exponentially for high T and as power law for $T = 0$)

Connection to quantum noise processes: Dissipative interactions

- Squeezed generalized amplitude damping channel: extends the concept of generalized amplitude damping channel by allowing for finite bath squeezing along with dissipation [R. Srikanth and SB: (2007)]
- It is characterized by the Kraus operators

$$
\begin{aligned}
E_0 &\equiv \sqrt{p_1}\begin{bmatrix} \sqrt{1-\alpha(t)} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1 \equiv \sqrt{p_1}\begin{bmatrix} 0 & 0 \\ \sqrt{\alpha(t)} & 0 \end{bmatrix}, \\
E_2 &\equiv \sqrt{p_2}\begin{bmatrix} \sqrt{1-\mu(t)} & 0 \\ 0 & \sqrt{1-\nu(t)} \end{bmatrix}, \\
E_3 &\equiv \sqrt{p_2}\begin{bmatrix} 0 & \sqrt{\nu(t)} \\ \sqrt{\mu(t)}e^{-i\Phi} & 0 \end{bmatrix}.
\end{aligned}
$$

Connection to quantum noise processes: Dissipative interactions

- Here

$$\begin{aligned}
\nu(t) &= \frac{N}{p_2(2N+1)}(1 - e^{-\gamma_0(2N+1)t}),\\
\mu(t) &= \frac{2N+1}{2p_2N}\frac{\sinh^2(\gamma_0 at/2)}{\sinh(\gamma_0(2N+1)t/2)}\exp\left(-\frac{\gamma_0}{2}(2N+1)t\right),\\
\alpha(t) &= \frac{1}{p_1}\left(1 - p_2[\mu(t)+\nu(t)] - e^{-\gamma_0(2N+1)t}\right),
\end{aligned}$$

where $p_2 = 1 - p_1$, and

$$\begin{aligned}
p_2 &= \frac{1}{(A+B-C-1)^2 - 4D}\\
&\times \left[A^2B + C^2 + A(B^2 - C - B(1+C) - D)\right.\\
&- (1+B)D - C(B+D-1)\\
&\pm 2\,(D(B - AB + (A-1)C + D)\\
&\times \left.(A - AB + (B-1)C + D))^{1/2}\right],
\end{aligned}$$

Connection to quantum noise processes: Dissipative interactions

- with

$$\begin{aligned} A &= \frac{2N+1}{2N}\frac{\sinh^2(\gamma_0 at/2)}{\sinh(\gamma_0(2N+1)t/2)}\exp\left(-\gamma_0(2N+1)t/2\right), \\ B &= \frac{N}{2N+1}(1-\exp(-\gamma_0(2N+1)t)), \\ C &= A+B+\exp(-\gamma_0(2N+1)t), \\ D &= \cosh^2(\gamma_0 at/2)\exp(-\gamma_0(2N+1)t). \end{aligned}$$

- If squeezing parameter r is set to zero, the Kraus operators reduce to that of a generalized amplitude damping channel, with $\nu(t)=\alpha(t)$, $\mu(t)=0$ and p_1 and p_2 becoming time-independent. If further $T=0$, then $p_2=0$, resulting in two Kraus operators, corresponding to an amplitude damping channel.

Dissipative Interaction with continued ...

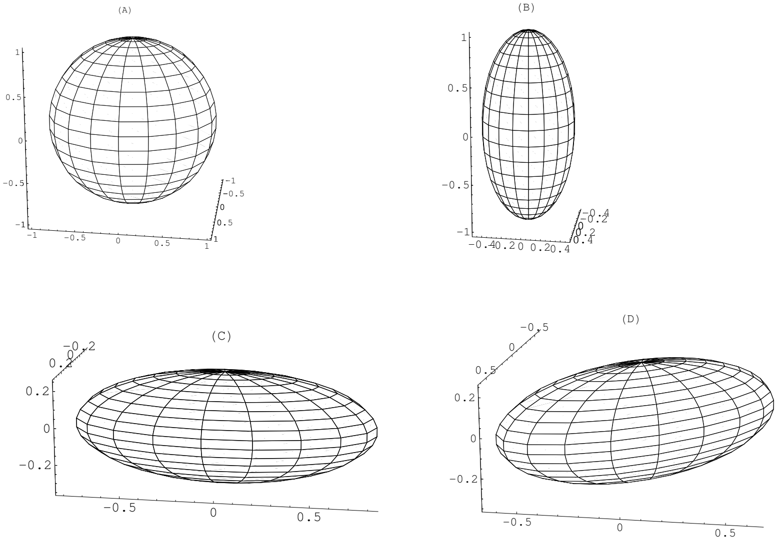


Fig. : Effect of QND and dissipative interactions on the Bloch sphere: (A) the full Bloch sphere; (B) the Bloch sphere after time $t = 20$, with $\gamma_0 = 0.2$, $T = 0$, $\omega = 1$, $\omega_c = 40\omega$ and the environmental squeezing parameter $r = a = 0.5$, evolved under a QND interaction ; (C) and (D) the effect of the Born-Markov type of dissipative interaction with $\gamma_0 = 0.6$ and temperature $T = 5$, on the Bloch sphere – the x and y axes are interchanged to present the effect of squeezing more clearly. (C) corresponds to $r = 0.4$, $\Phi = 0$ and $t = 0.15$ while (D) corresponds to $r = 0.4$, $\Phi = 1.5$ and $t = 0.15$.
[SB and R. Ghosh: (2007)]

- We now make a brief excursion into non-Markovian Open Quantum Systems [Rivas, Huelga, Plenio (2014); Breuer,Laine,Piilo,Vacchini (2016); Vega,Alonso (2017)].
- This is a bigger class than the Markovian ones discussed till now.
- We will illustrate our discussions using three useful models: Random Telegraph Noise (RTN), modified Ornstein-Uhlenbeck and Power Law noises.
- Some promiment diagnostics of non-Markovian behavior, developed in recent years, are breifly discussed.

- Consider the time dependent stochastic Hamiltonian describing the evolution of a single qubit [Daffer, Wodkiewicz, Cresser,McIver (2004)]

$$H(t) = \hbar \sum_{i=1}^{3} V_i(t)\sigma_i,$$

where σ_i denote Pauli matrices and $V_i(t) = a_i(-1)^{n_i(t)}$ is the representation of the Random Telegraph Noise signal (RTN).

- RTN is a non-Gaussian stationary stochastic process that fluctuates randomly between binary amplitude values $\pm a$, following the Poisson probability distribution realized by the random variable $n_i(t)$.
- Autocorrelation function for RTN $V_i(t)$ is

$$\langle V_i(t)V_j(t')\rangle = \delta_{ij}a^2 e^{-|t-t'|/\tau},$$

- Example, a two-level atom driven by a laser source with rapidly varying phase noise.

- a has the significance of the strength of the system-environment coupling and $1/\tau$ is proportional to the fluctuation rate of the RTN. The Fourier transform of the correlation function results in a Lorentzian power spectral density with peak value given by $2a^2\tau$.
- The Kraus operators representing the process are

$$\begin{aligned} K_1 &= \sqrt{1+\Lambda(\nu)/2}I, \\ K_2 &= \sqrt{1-\Lambda(\nu)/2}\sigma_3, \end{aligned}$$

satisfying the completeness relation $\sum_{n=1}^{2} K_n^\dagger K_n = I$.

- $\Lambda(\nu)$ represents the damped harmonic function which encodes both the Markovian and non-Markovian behavior of the qubit,

$$\Lambda(\nu) = e^{-\nu}[\cos(\nu\mu) + \sin(\nu\mu)/\mu],$$

where $\mu = \sqrt{(\frac{2a}{\gamma})^2 - 1}$ is the frequency of the harmonic oscillators and $\nu = \gamma t$ is the dimensionless time.

- γ is the fluctuation rate and is equal to $1/2\tau$.
- The function $\Lambda(\nu)$ corresponds to two regimes; the purely damping regime, where $a\tau < 0.25$, and damped oscillations for $a\tau > 0.25$. Corresponding to these two regimes of $\Lambda(\nu)$, we observe Markovian and non-Markovian behavior, respectively.

- OUN: a non-Markovian process with a well defined Markov limit [Uhlenbeck,Ornstein (1930)].
- Example: Spin of an electron interacts with a magnetic field subject to stochastic fluctuations.
- The effective hamiltonian for such an interaction

$$H_{tot}(t) = \frac{\Omega(t)}{2}\sigma_z$$

- $\Omega(t)$ is a random variable that represents the fluctuations of the spin transition frequencies. The statistical mean values associated with $\Omega(t)$,

$$\begin{aligned} M[\Omega(t)] &= 0, \\ M[\Omega(t),\Omega(s)] &= \Gamma\gamma e^{-\gamma|t-s|} \end{aligned}$$

- γ specifies the noise bandwidth and $\gamma^{-1} = \tau_c$ is the finite correlation time of the environment. The parameter Γ is the effective relaxation time.

- A Krauss operator-sum-representation is [Yu,Eberley (2010)]

$$
\begin{aligned}
K_1 &= |0\rangle\langle 0| + p\,|1\rangle\langle 1| \\
K_2 &= \sqrt{1-p^2}\,|1\rangle\langle 1|
\end{aligned}
$$

- $p \equiv p(t) = \exp[-\frac{\Gamma}{2}\{t + \frac{1}{\gamma}(e^{-\gamma t} - 1)\}]$.

Non-Markovian Aspects:Power Law Noise (PLN)

- PLN, also called $1/f^{\alpha}$ noise is a non-Markovian stationary noise process, where α is some real number.
- The PLN is a major source of decoherence in solid state quantum information processing devices such as superconducting qubits [Paladino *et.al.* (2014)].
- The statistical properties such as the mean and auto-correlation function associated with the PLN processes are,

$$\begin{aligned} M[\Omega(t)] &= 0, \\ M[\Omega(t),\Omega(s)] &= \tfrac{1}{2}(\alpha-1)\alpha\Gamma\frac{1}{(\gamma|t-s|+1)^{\alpha}} \end{aligned}$$

  where, the parameters γ and Γ are as in OUN.
- The Kraus operators representing the dynamical map for $\alpha = 3$ are given by,

$$\begin{aligned} K_1 &= |0\rangle\langle 0| + q\,|1\rangle\langle 1| \\ K_2 &= \sqrt{1-q^2}\,|1\rangle\langle 1| \end{aligned}$$

- $q \equiv q(t) = \exp(-\frac{0.5t(t\gamma+2)\Gamma\gamma}{(t\gamma+1)^2})$.

- A family of dynamical maps $\Phi(t,0)$ is defined to be divisible if for all $t_2 \geq t_1 \geq 0$ there exists a CPT map $\Phi(t_2,t_1)$ such that the relation $\Phi(t_2,0) = \Phi(t_2,t_1)\Phi(t_1,0)$ holds.
- The simplest example of a divisible quantum process is given by a dynamical semigroup. For a semigroup $\Phi(t,0) = \exp[\mathcal{L}t]$ and divisibility is satisfied with the CPT map $\Phi(t_2,t_1) = \exp[\mathcal{L}(t_2 - t_1)]$.
- Consider now a quantum process given by the time-local master equation with a time dependent generator. The dynamical maps can then be represented in terms of a time-ordered exponential,

$$\Phi(t,0) = \mathrm{T}\exp\left[\int_0^t dt'\mathcal{K}(t')\right], \quad t \geq 0,$$

where T denotes the chronological time-ordering operator.

- We can also define the maps

$$\Phi(t_2, t_1) = \mathrm{T}\exp\left[\int_{t_1}^{t_2} dt'\mathcal{K}(t')\right], \quad t_2 \geq t_1 \geq 0,$$

  such that the composition law $\Phi(t_2, 0) = \Phi(t_2, t_1)\Phi(t_1, 0)$ holds by construction. The maps $\Phi(t_2, t_1)$ are completely positive, as is required by the divisibility condition, if and only if the decay rates $\gamma_i(t)$ of the generator are positive functions. Thus divisibility is equivalent to positive rates in the time-local master equation [Laine et al. (2010)].
- Thus non-Markovian quantum processes could be described by time-local master equations whose generator involves at least one temporarily negative rate $\gamma_i(t)$ [Hall, Cresser, Li and Andersson (2014)].

- In this context a characterization of non-Markovianity was given by [Rivas, Huelga, Plenio (2010)]

$$f(t) = \lim_{\epsilon \to 0^+} \frac{|| \left[\Phi(\mathrm{t} + \epsilon, \mathrm{t}) \otimes \mathcal{I} \right] \left(|\Psi\rangle\langle\Psi| \right) ||_1 - 1}{\epsilon} \tag{1}$$

- Follows from Choi that $f(t) > 0$ for non-Markovian evolution.
- Measure $\mathcal{M} = \int_I dt f(t)$, non-Markovian behavior in $t \in I$.

- Consider two parties, Alice and Bob. Alice prepares a quantum system in one of two states ρ^1 or ρ^2 with probability $\frac{1}{2}$ each, and then sends the system to Bob. It is Bob's task to find out by a single measurement on the system whether the system state was ρ^1 or ρ^2. It turns out that Bob cannot always distinguish the states with certainty, but there is an optimal strategy which allows him to achieve the maximal possible success probability given by

$$P_{\text{max}} = \frac{1}{2}\left[1 + D(\rho^1, \rho^2)\right].$$

- The trace distance $D(\rho^1, \rho^2) = \frac{1}{2}||\rho^1 - \rho^2|| = \frac{1}{2}\mathrm{tr}|\rho^1 - \rho^2|$ can therefore be interpreted as a measure for the distinguishability of the quantum states ρ^1 and ρ^2. Here $\mathrm{tr}|A| = \mathrm{tr}\sqrt{A^\dagger A}$.

- The trace distance between any pair of states satisfies $0 \leq D(\rho^1, \rho^2) \leq 1$.
- The trace distance is sub-additive with respect to tensor products of states

$$D(\rho^1 \otimes \sigma^1, \rho^2 \otimes \sigma^2) \leq D(\rho^1, \rho^2) + D(\sigma^1, \sigma^2).$$

- The trace distance is invariant under unitary transformations U,

$$D(U\rho^1 U^\dagger, U\rho^2 U^\dagger) = D(\rho^1, \rho^2).$$

More generally, all trace preserving and completely positive maps, i.e., all trace preserving quantum operations Λ are contractions of the trace distance,

$$D(\Lambda\rho^1, \Lambda\rho^2) \leq D(\rho^1, \rho^2).$$

- No quantum process describable by a family of CPT dynamical maps can ever increase the distinguishability of a pair of states over its initial value.
- When a quantum process reduces the distinguishability of states, information is flowing from the system to the environment. Correspondingly, an increase of the distinguishability signifies that information flows from the environment back to the system.
- The definition for quantum non-Markovianity, discussed here, is based on the idea that for Markovian processes any two quantum states become less and less distinguishable under the dynamics, leading to a perpetual loss of information into the environment.
- Quantum memory effects thus arise if there is a temporal flow of information from the environment to the system. The information flowing back from the environment allows the earlier open system states to have an effect on the later dynamics of the system, which implies the emergence of memory effects [Breuer et al. (2009)].

- A quantum process described in terms of a family of quantum dynamical maps $\Phi(t,0)$ is non-Markovian if there is a pair of initial states $\rho_S^{1,2}(0)$ such that the trace distance between the corresponding states $\rho_S^{1,2}(t)$ increases at a certain time $t > 0$:

$$\sigma(t, \rho_S^{1,2}(0)) \equiv \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) > 0,$$

  where $\sigma(t, \rho_S^{1,2}(0))$ denotes the rate of change of the trace distance at time t corresponding to the initial pair of states.
- This implies that the class of quantum dynamical semigroups are Markovian.

- This suggests defining a measure $\mathcal{N}(\Phi)$ for the non-Markovianity of a quantum process through [Breuer et al. (2009)]

$$\mathcal{N}(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma > 0} dt\, \sigma(t, \rho_S^{1,2}(0)).$$

- The time integration is extended over all time intervals (a_i, b_i) in which σ is positive and the maximum is taken over all pairs of initial states. The measure can be written as

$$\mathcal{N}(\Phi) = \max_{\rho^{1,2}(0)} \sum_i \left[D(\rho_S^1(b_i), \rho_S^2(b_i)) - D(\rho_S^1(a_i), \rho_S^2(a_i)) \right].$$

To calculate this quantity one first determines for any pair of initial states the total growth of the trace distance over each time interval (a_i, b_i) and sums up the contribution of all intervals. $\mathcal{N}(\Phi)$ is then obtained by determining the maximum over all pairs of initial states.

Discrete Time Quantum Walk (DTQW) in 1D

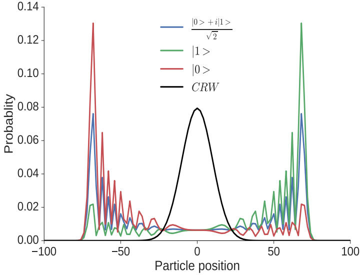
- Walk is defined on $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$.
  $\mathcal{H}_c$ (particle/coin) is spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$.
  $\mathcal{H}_p$ (position) is spanned by $|x\rangle$, $x \in \mathbb{Z}$.
- Initial State: $|\psi_{in}\rangle = \left[\cos(\delta)|\uparrow\rangle + e^{i\eta}\sin(\delta)|\downarrow\rangle\right] \otimes |x=0\rangle$
- Evolution:
  - Coin operation: Hadamard operation H:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

  - Conditional unitary shift operation S

$$\begin{aligned} S &= |\uparrow\rangle\langle\uparrow| \otimes \sum_{x\in\mathbb{Z}} |x-1\rangle\langle x| + |\downarrow\rangle\langle\downarrow| \otimes \sum_{x\in\mathbb{Z}} |x+1\rangle\langle x| \\ &\equiv |\uparrow\rangle\langle\uparrow| \otimes \hat{A} + |\downarrow\rangle\langle\downarrow \otimes \hat{A}^\dagger. \end{aligned}$$

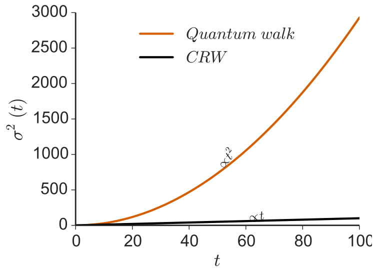
  - $\hat{A}$ and $\hat{A}^\dagger$ are unitary operators.
- Each QW step (Hadamard walk): $W = S\big(H \otimes I\big)$.



100 step of Classical Random Walk (CRW) and QW $\left[S(H \otimes I)\right]^{100}$ on a particle starting from the initial state $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$

- G. V. Riazanov (1958), R. P. Feynman (1986)
- Aharonov, Davidovich, Zugury (1993)
- Chandrashekar, Srikanth, SB (2007)

Variance of DTQW

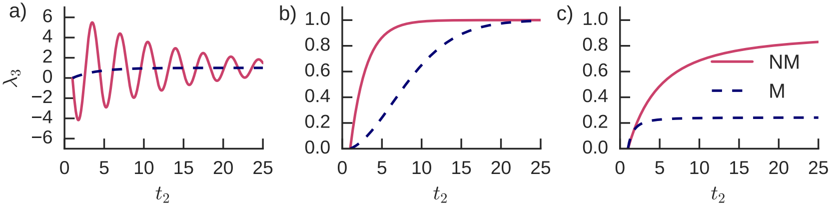


Variance, measure of how much the walker has spread from the origin

$$\sigma^2 = \sum_{i=1}^{n} p_i (i - \mu)^2$$

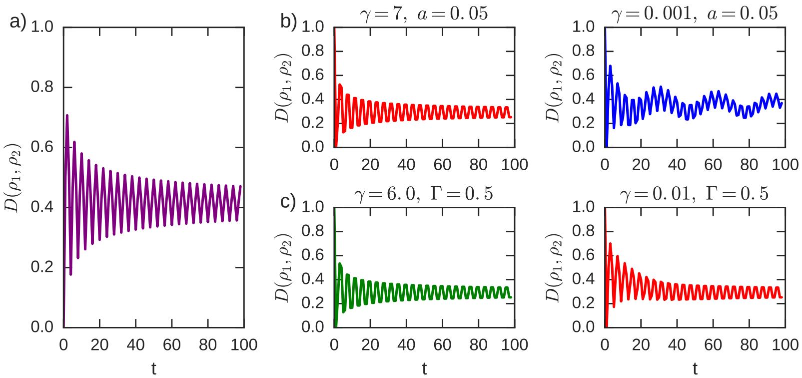
p_i is the probability of finding the walker at the i_{th} position and $\mu = \sum_i p_i x_i$, as a function of time for 100 steps of Hadamard quantum walk (red line) and classical random walk (black solid line)

- In Markovian open-system dynamics Λ given two distinct states ρ and σ, distance measures $\mathfrak{D}$ (such as relative entropy or trace distance (TD)) satisfy $\mathfrak{D}[\Lambda(\rho), \Lambda(\sigma)] \leq \mathfrak{D}[\rho, \sigma]$
- Closeness measures $\mathfrak{C}$ (such as fidelity or mutual information (MI)) satisfy $\mathfrak{C}[\Lambda(\rho), \Lambda(\sigma)] \geq \mathfrak{C}[\rho, \sigma]$.
- By contrast, non-Markovian dynamics can violate this monotonicity property... could be attributed to information backflow from the environment.



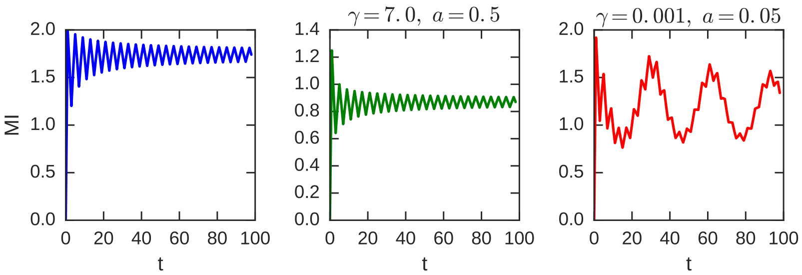
The eigen value of the Choi matrix obtained from intermediate dynamical map in both the Markovian and non-Markovian regimes for different noise models [P. Kumar, SB, R Srikanth, V. Jagadish, F. Petruccione, (2017)]. The inital time $t_1 = 1$ is fixed for all the plots (a) RTN: In the non-Markovian regime ($\gamma = 0.05, a = 0.9$) we observe λ_3 oscillating between positive and negative eigen values and in the Markovian regime ($\gamma = 5, a = 0.9$) λ_3 is always positive. (b) OUN: In both the non-Markovian ($\gamma = 0.05, \Gamma = 1.0$) and the Markovian regime ($\gamma = 5.0, \Gamma = 1.0$) λ_3 is positive. (c) PLN: Similar to OUN, no negative value for for λ_3 is ovserved in both the non-Markovian regime ($\gamma = 0.05, \Gamma = 5.0$), Markovian regime ($\gamma = 2.0, \Gamma = 5.0$).

TD of DTQW under non-Markov evolution



TD evolution with respect to number of time steps t, under the influence of different non-Markovian noise sources. (a) RTN: The noiseless quantum walk, i.e., the QW in the absence of an external noise, shows rapid recurrences due to interaction with the position "environment" [Hinarejos, et al. (2014)], (b) RTN: In the Markovian regime (middle) damping of the position induced oscillations is observed, while in the non-Markovian regime (right) an additional oscillatory frequency component is present due to RTN-induced decoherence. (c) OUN: In the Non-Markovian regime (right) TD decays without the additional oscillatory feature in contrast to RTN case, indicating lack of backflow due to noise. PLN, like OUN, does not manifest backflow.

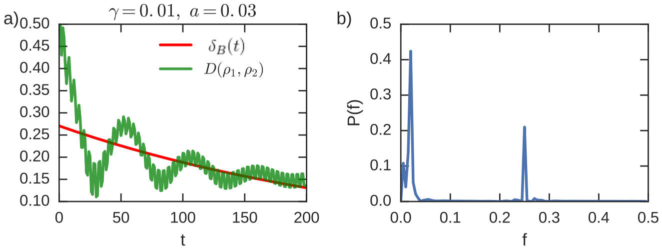
MI of DTQW under non-Markov evolution



MI evolution with respect to number of time steps t, under the influence of RTN noise. The Noiseless case of the walk in the left figure shows oscillations due to position environment. The center plot is the Markovian regime of RTN with amplitude of oscillations reduced due to the noise. The right figure is the non-Markovian regime of RTN which features oscillations at two different frequencies, one due to position and the other due to the low-frequency backflow component due to the RTN . Both OUN and PLN does not feature this low-frequency component, indicating the absence of backflow in these noise models.

Disambiguating different sources of non-Markovianity: Power spectral analysis

- In addition to the oscillations present in the noiseless evolution of quantum walk, a further recurrence structure due to backflow arises when the coin is exposed to an external noise such as RTN in the (strong) non-Markovian regime.
- To disambiguate these two distinct sources (position and the bona fide environment) of non-Markovian backflow, we compute the power spectrum of the "trend-corrected time evolution" of TD, i.e., $\Delta(t) \equiv D(\rho_1(t), \rho_2(t)) - \delta_B(t)$, where $\delta_B(t)$ is the monotonically falling best fit (MFBF) to $D(\rho_1(t), \rho_2(t))$.
- The fit is quantified according to a suitable distance measure. If the evolution is Markovian, then $D(\rho_1(t), \rho_2(t))$ falls monotonically, and thus $D(\rho_1(t), \rho_2(t)) = \delta_B(t)$.
- $\Delta(t)$ is like a measure of non-Markovianity, since if $\Delta(t) = 0$ then backflow is absent.
- Power spectral analysis allows to locate in the frequency domain the different sources of non-Markovian backflow. In this case, the position degree of freedom serves as one source, while RTN producing environment serves as the other source. In general, multiple sources of non-Markovian backflow may be identified in a complex system.
- Power spectrum method involves the following procedure: function f_t is computed for time $t \leq N = 100$ steps. The absolute squared of the functions Discrete Fourier Transform (DFT) is then computed, i.e., $\mathcal{S}(k) = |\sum_{n=0}^{N-1} f_n e^{-2\pi ikn/N}|^2$, where f_n can be the trend-corrected TD or MI.



Filtering and power spectral analysis of TD between coin states driven by RTN [SB, P. Kumar, R Srikanth, V. Jagadish, F. Petruccione, (2017)]. Power spectrum analysis of TD (Trace Distance) under RTN. (a) TD evolution under RTN and its MFBF (δ_B), appears to approximate an exponential fall. (b) The power spectrum of the filtered plot $\Delta(t)$. The power at $f \approx 0.27$ (resp., $f \approx 0.03$) corresponds to the position (resp., RTN environment). This filtering approach allows us to compare the relative strengths of the two sources of non-Markovianity (the area under position environment is 1.5 times more). In OUN or PLN, no frequency component associated with the noisy environment is seen, given their lack of backflow.

- Time evolution of QW induces non-classical correlations between coin and position degrees of freedom.

Concurrence

For a mixed state ρ of two qubits, concurrence [Wootters (1998)], which is a measure of entanglement, is $C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$. λ_i are the square root of the eigenvalues, in decreasing order, of the matrix $\rho^{\frac{1}{2}}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\rho^{\frac{1}{2}}$ where ρ is computed in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
For a two-qubit system, concurrence is equivalent to the entanglement of formation which can then be expressed as a monotonic function of concurrence C as
$E_F = -\frac{1+\sqrt{1-C^2}}{2}\log_2(\frac{1+\sqrt{1-C^2}}{2}) - \frac{1-\sqrt{1-C^2}}{2}\log_2(\frac{1-\sqrt{1-C^2}}{2})$.

Measurement induced disturbance (QMID)

- QMID quantifies the quantumness of the correlation between the quantum bipartite states shared amongst Alice and Rob.
- For the given $\rho'_{A,R}$, if ρ'_A and ρ'_R are the reduced density matrices, then the mutual information that quantifies the correlation between Alice and Rob is

$$I = S(\rho'_A) + S(\rho'_R) - S(\rho'_{A,R}),$$

  $S(.)$ is the von Neumann entropy.
- If $\rho'_A = \sum_i \lambda^i_A \Pi^i_A$ and $\rho'_R = \sum_j \lambda^j_R \Pi^j_R$ denotes the spectral decomposition of ρ'_A and ρ'_R, respectively, then the state $\rho'_{A,R}$ after measuring in joint basis $\{\Pi_A, \Pi_R\}$ is

$$\Pi(\rho'_{A,R}) = \sum_{i,j} (\Pi^i_A \otimes \Pi^j_R) \rho'_{A,R} (\Pi^i_A \otimes \Pi^j_R).$$

- QMID [Luo (2008)] is

$$M(\rho'_{A,R}) = I(\rho'_{A,R}) - I(\Pi(\rho'_{A,R}))$$

  is a measure of quantumness of the correlation.

Discord

- Quantum discord (QD)[Ollivier and Zurek (2001), Henderson and Vedral (2001)] is used to estimate the difference between the total and classical correlations during the course of QW evolution.
- It is the difference between two natural extensions of classical mutual information to the quantum setting. The first definition of MI is the standard one and the second definition is the quantum version of conditional entropy which depends on the measurement process. It is defined as

$$\mathcal{J}(\rho_P|\rho_C) = \mathcal{S}(\rho_P) - \mathcal{S}(\rho_P|\Pi_i^C)$$

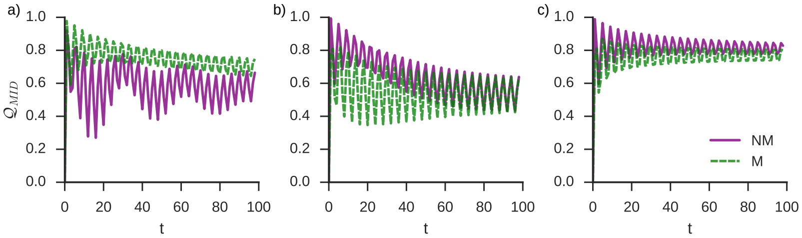
  where $\mathcal{S}(\rho_P|\Pi_i^C)$ is the quantum conditional entropy, defined with respect to a set of projective measurements $\{\Pi_i\}$ performed on the coin state ρ_C.
- QD is

$$\mathcal{D}(\rho_P|\rho_C) = \mathcal{I}(\rho_P : \rho_C) - \max_{\Pi_i^C} \mathcal{J}(\rho_P|\rho_C).$$

- QD can be thought of as an optimal version of MID.
- In MID, the measurement operators that characterize the quantum correlations between the coin and position are given by the corresponding spectral projections and does not involve any optimization. Hence, MID is an operational measure in the sense that computing it is relatively simple. QD on the other hand, optimizes over all possible projectors to compute the correlation between the coin and position states, which makes it computationally intensive.

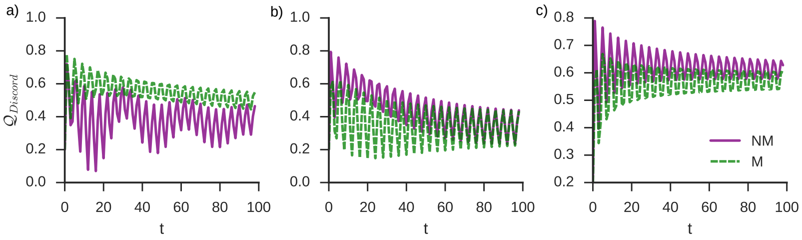
Quantum purity

- Quantum state purity is another interesting quantity that can be used to study the quantumness of quantum walks.
- Purity is a monotone under Markovian evolutions. However under non-Markovian evolutions temporary revivals in purity can be observed in the form of oscillations, this idea has been exploited and studied as a measure of non-Markovianity [S. Bhattacharya, A. Misra, C. Mukhopadhyay, and A. K. Pati, (2017)].
- Purity can be studied using Von-Neumann entropy $S(\rho_c) = -tr(\rho_c \ln \rho_c)$ where ρ_c is the reduced state of the particle obtained by tracing out the position degrees of freedom. Note that von-Neumann entropy is maximum for a completely mixed state for which the purity is zero.

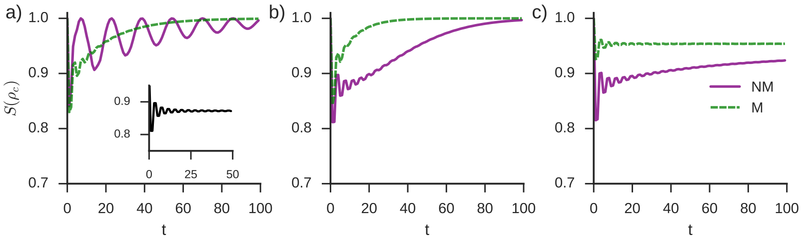


MID for 100 steps of quantum walk in the presence of RTN, OUN and PLN. (a) RTN: The interaction between coin and RTN induces oscillations in the non-Markovian regime indicating the backflow of information from the environment. In the Markovian regime backflow from RTN is absent and the oscillation is due to position induced non-Markovianity. (b) OUN: In the Non-Markovian region unlike RTN no backlow due to the noise is observed, however the decay rate is lesser compared to the Markovian regime. (C). PLN is similar to OUN with no backflow in the non-Markovian regime.

Quantumness of quantum walk: QD



QD as a function of walk steps t, under the influence of RTN, OUN and PLN. The results are similar to MID for all three noise sources (a) RTN, (b) OUN and (c) PLN, respectively, that drive the coin. QD is strictly a lower bound on the MID between the coin and the position states, due to the optimization performed over the measurement operators.



Von-Neumann entropy under different sources of non-Markovianity. (a) RTN: The backflow effect in the form of oscillations is clearly seen in the non-Markovian regime. The inset plot is the von-Neumann entropy of noiseless quantum walk. (b) OUN: In both non-Markovian and Markovian regimes only the position induced non-Markovianity can be observed. (C) PLN: Similar to OUN, the backflow effects are absent in PLN.

Conclusion

- We talked about some recent developments in the efforts to understand non-Markovian phenomenon.
- Our discussion about non-Markovian behaviour was made in the backdrop of a few well-known non-Markovian processes.
- We show how different sources of non-Markovianity could be disambiguated.
- This was done using the platform of Quantum Walks.