Rigorous Quantum Limits on Monitoring Free Masses and Oscillators.

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Abstract.

For a harmonic oscillator of mass m and frequency ω, I derive the rigorous quantum limits,

$$|\sigma^2(X(t)) - \sigma^2(X(0))\cos^2(\omega t) \\ -\sigma^2(P(0))\sin^2(\omega t)/(m\omega)^2| \\ \leq |\sin(2\omega t)/(2m\omega)|\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2},$$

and the 'maximally contractive' and 'maximally expanding' states which saturate them.Here $\sigma^2(X(t))$ and $\sigma^2(P(t))$ are the variances of the Heisenberg position and momentum operators at time t . The bounds and extremal contractive states for the oscillator and for a free mass (also derived here) are useful for gravitational wave detection and optical communication ; contractive states for oscillators improve on the Schrödinger coherent states of constant variance.

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Introduction. A quantum system is prepared, for example by a measurement, in an initial state . Subsequent monitoring or measurements of an observable A may be useful to detect any external disturbances additional to the intrinsic change in the uncertainty of the observable due to the system evolving by its own Hamiltonian.Much before the actual discovery of gravitational waves [0] it was realised that accurate monitoring of position of an oscillator and of a free mass, including quantum effects are important for gravitational wave interferometers [0].

For an arbitrary initial state of a free mass or an oscillator , I shall obtain rigorous quantum limits (RQL) on the intrinsic uncertainty after time t.

For any observable with Schrödinger operator A (e.g. position $A = X$ or momentum $A = P$), and any Hamiltonian H, the Heisenberg operator $A(t)$ at time t and its variance $\sigma^2(A(t))$ are defined by,

$$A(t) \equiv exp(iHt/\hbar)\, A\, exp(-iHt/\hbar), \quad (1)$$

$$\sigma^2(A(t)) \equiv < \psi(0)|(\Delta A(t))^2|\psi(0) >, \quad (2)$$

$$\Delta A(t) \equiv A(t) - < A(t) >, \quad (3)$$

$$< A(t) > \equiv < \psi(0)|A(t)|\psi(0) > \quad (4)$$

where $|\psi(0)>$ is the initial state.

Heuristic Standard Quantum Limit on Monitoring Position of a Free Mass . There are heuristic arguments proposing that the accuracy of position monitoring is limited by the standard quantum limit (SQL) ([0],[0]) on the variance of the position operator $X(t)$ due to back reaction of the momentum uncertainty :

$$\sigma^2(X(t)) \geq \sigma^2(X(0)) + (t^2/m^2)\sigma^2(P(0)) \\ \geq 2(t/m)\sigma(X(0))\sigma(P(0)) \geq \hbar t/m, \quad (5)$$

For the free mass , $H = P^2/(2m)$. the inequality (5) is actually an equality for Gaussian states,

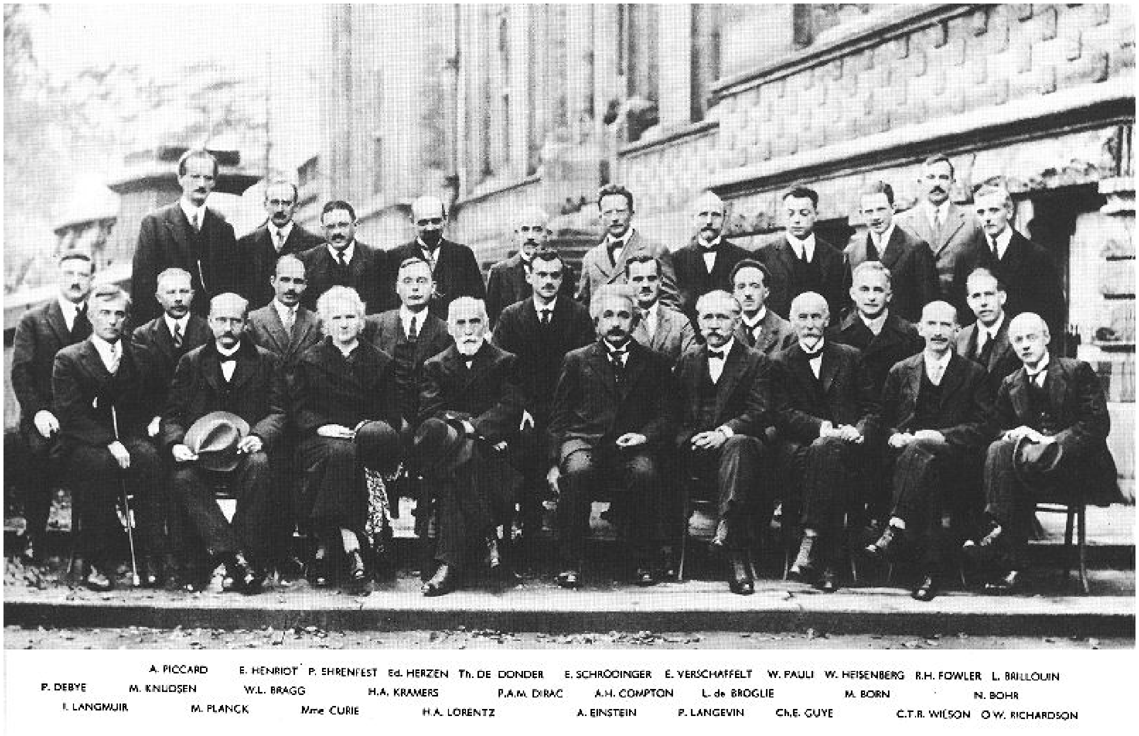
$$< p|\psi(t) >= (\pi\alpha)^{-1/4} exp[-\frac{(p-\beta)^2}{2\alpha} - it\frac{p^2}{2m}],$$
$$\sigma^2(P(t)) = \frac{\alpha}{2}, \ \sigma^2(X(t)) = \hbar^2\frac{1+(\alpha t/(m\hbar))^2}{2\alpha}. \quad (6)$$

$$< x|\psi(t) >= \frac{(\alpha/\pi)^{1/4}}{\sqrt{1 + it\alpha/(m\hbar)}} exp(f),$$

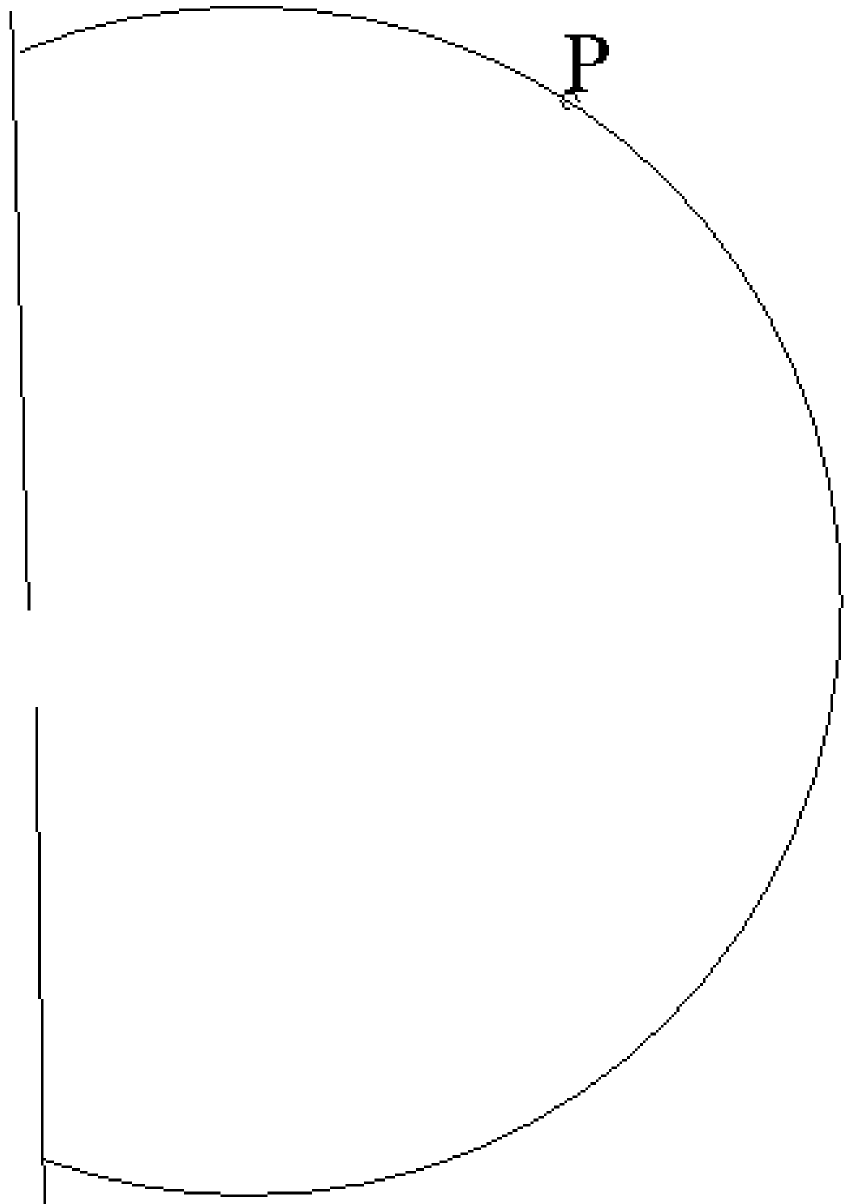
$$f = -\frac{\alpha}{2\hbar^2\Big(1 + t^2(\alpha/(m\hbar))^2\Big)} \times$$

$$\Big[(x - \frac{\beta t}{m})^2 - it\frac{\alpha}{m\hbar}(x^2 - (\hbar\beta/\alpha)^2) - 2i\hbar\beta x/\alpha\Big] \quad (7)$$

A small position uncertainty at $t = 0$ (i.e. large α) forces a rapid expansion of the wave packet size with time. This was the basis of Einstein questioning the standard interpretation of quantum mechanics at the 1927 Solvay conference.



At the 1927 Solvay conference Einstein discussed the example of a particle passing through a narrow hole on to a hemispherical fluorescent screen which records the arrival of the particle.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Suppose that a scintillation is seen at a point P at time $t = T$, and suppose that the hole is so narrow that the wave packet corresponding to the particle is uniformly spread all over the screen at t slightly less than T. Was the particle somewhere near P at $t = T - \epsilon$ (ϵ small)? Ordinary quantum mechanics says that the probabilities at $t = T - \epsilon$ for the particle being anywhere on the screen are uniform (and not particularly large in the vicinity of P). Thus the naive history corresponding to the idea of a particle with a trajectory (<u>any</u> trajectory) is denied.

The SQL tries to argue that this problem is generic, not limited to the Gaussian wave packet. One heuristic argument for the SQL ([0],[0]) Eq. (5) starts from

$H = P^2/(2m), \Delta X(t) = \Delta X(0) + (t/m)\Delta P(0),$

and hence,

$$\sigma^2(X(t)) = \sigma^2(X(0)) + (t^2/m^2)\sigma^2(P(0)) + \frac{t}{m} < \psi(0)|\Delta X(0)\Delta P(0) + \Delta P(0)\Delta X(0)|\psi(0) > \quad (8)$$

One obtains the SQL if one assumes that the third term on the right-hand side is non-negative.

In a seminal paper, Yuen [0] noted that there are contractive states for which this assumption is incorrect. In an interesting and correct argument for the SQL, Caves [0] noted that in certain measurement models, resolution of the meter $\geq \sigma(X(0))$ may entail that the variance of the position measurement at time t is $\geq \sigma^2(X(0)) + \sigma^2(X(t))$ which is $\geq \hbar t/m$ by the uncertainty principle. Yuen [0] and Ozawa [0] (see also [0]), point out the existence of other measurement models for which the imperfect resolution correction can be much smaller than $\sigma^2(X(0))$.

At first, I address myself to finding a rigorous version of the heuristic SQL Eq. (5) on $\sigma^2(X(t))$ and optimum contractive staes. Later I present an explicit Arthurs-Kelly type ([0],[0]) realisation of measurements for accurate monitoring of free masses and harmonic oscillators using contractive meter states.

Rigorous Quantum Limit on Monitoring Position of a Free Mass . We start from Eq. (8) and find exact limits on the third term on the right-hand side. Using

$$[\Delta X(0), \Delta P(0)] = i\hbar, \tag{9}$$

we have,

$$\begin{aligned} < & \psi(0)|\Delta X(0)\Delta P(0)+\Delta P(0)\Delta X(0)|\psi(0)> + i\hbar \\ = & \; 2 < \psi(0)|\Delta X(0)\Delta P(0)|\psi(0)> . \end{aligned} \tag{10}$$

Schwarz inequality on the right-hand side yields,

$$\begin{aligned} \Big(& < \psi(0)|\Delta X(0)\Delta P(0)+\Delta P(0)\Delta X(0)|\psi(0)> \Big)^2 \\ \leq & \; 4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2. \end{aligned} \tag{11}$$

Substituting this into Eq. (8) I have the rigorous quantum limits (RQL),

$$\sigma^2(X(0)) + (t/m)^2\sigma^2(P(0))$$
$$-(t/m)\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2} \leq \sigma^2(X(t))$$
$$\leq \sigma^2(X(0)) + (t/m)^2\sigma^2(P(0))$$
$$+(t/m)\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}. \qquad (12)$$

It must be stressed that the bounds are fundamental quantum limits valid for arbitrary states. The only states saturating the inequalities are those for which the Schwarz inequalities are equalities, i.e. $\Delta P(0)|\psi(0)>$ is a complex constant times $\Delta X(0)|\psi(0)>$.

Hence the RQL ,Eq. (12) are equalities if and only if,

$$< X'|\psi(0) >= \Big(\frac{Re\lambda}{\pi\hbar}\Big)^{1/4}$$
$$exp\Big(\frac{i < P(0) > X'}{\hbar} - \frac{\lambda(X'- < X(0) >)^2}{2\hbar}\Big), \tag{13}$$

with

$$Im\lambda = \pm\frac{1}{2\sigma^2(X(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2},$$
$$Re\lambda = \hbar/(2\sigma^2(X(0))), \tag{14}$$

with the positive and negative signs of $Im\lambda$

corresponding respectively to saturation of the left-hand side and right-hand side of the inequality(12). The right-hand side of inequality (12) sets an upper limit on spreading of the position wave packet and the left-hand side to the amount of contraction possible. The states saturating the left-hand side of inequality(12) are essentially Yuen's contractive Twisted Coherent States (TCS) [0],thus demonstrating their optimality for given $\sigma(X(0)), \sigma(P(0))$. It is useful to rewrite this inequality in two alternative forms:

$$\sigma^2(X(t)) \geq \Big(\frac{\hbar}{2\sigma(P(0))}\Big)^2 + \Big(\frac{\sigma(P(0))}{m}\Big)^2 (t - t_m)^2 \tag{15}$$

$$= \frac{t}{m}\Big(2\sigma(X(0))\sigma(P(0)) - \sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}\Big) + \Big(t\frac{\sigma(P(0))}{m} - \sigma(X(0)\Big)^2, \tag{16}$$

where ,

$$t_m = \frac{m}{2\sigma^2(P(0))}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}. \tag{17}$$

Eq.(15) shows that the optimal state remains contractive upto time t_m, and the variance $\sigma^2(X(t))$ is less than the initial variance $\sigma^2(X(0))$ for time $t < 2t_m$.; Eq. (16) shows that for a given uncertainty product, by choosing $(t/m)\sigma^2(P(0)) = \sigma(X(0))\sigma(P(0))$, $\sigma^2(X(t))$ can be made as small as

$$(t/m)\Big(2(\sigma(X(0))\sigma(P(0)) - \sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}\Big)$$

which can be much smaller than the heuristic standard quantum limit for a large uncertainty product.

Rigorous Quantum Limits on Monitoring Position or Momentum of a Harmonic Oscillator . This problem is specially significant because Hamiltonians for all free Bosonic fields , including the electromagnetic field , are sums of Harmonic oscillator Hamiltonians. In particular, the limits I derive can be immediately translated into rigorous quantum limits (RQL) on time development of quadratures of the electromagnetic field.

The Hamiltonian $H = P^2/(2m) + \frac{1}{2}m\omega^2 X^2$

can be rewritten as,

$$H = \frac{1}{2}\hbar\omega(p^2 + x^2) = \hbar\omega(a^\dagger a + 1/2), \quad (18)$$

where,

$$p = \frac{P}{\sqrt{m\hbar\omega}},\ x = \sqrt{\frac{m\omega}{\hbar}}X,$$
$$a = \frac{x + ip}{\sqrt{2}},\ a^\dagger = \frac{x - ip}{\sqrt{2}}. \quad (19)$$

The Heisenberg equations of motion yield,

$$\begin{aligned} \Delta x(t) &= \cos(\omega t)\, \Delta x(0) + \sin(\omega t)\, \Delta p(0) \\ \Delta p(t) &= -\sin(\omega t)\, \Delta x(0) + \cos(\omega t)\, \Delta p(0). \end{aligned}$$

Hence,

$$\sigma^2(x(t)) = \cos^2(\omega t)\sigma^2(x(0)) + \sin^2(\omega t)\sigma^2(p(0)) + \frac{1}{2}\sin(2\omega t) < \psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0) >,$$

$$\sigma^2(p(t)) = \sin^2(\omega t)\sigma^2(x(0)) + \cos^2(\omega t)\sigma^2(p(0)) - \frac{1}{2}\sin(2\omega t) < \psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0) >($$

As before, using $[\Delta x(0), \Delta p(0)] = i$, and Schwarz inequality, we obtain,

$$\Big(< \psi(0)|\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)|\psi(0) > \Big)^2 \leq 4\sigma^2(x(0))\sigma^2(p(0)) - 1. \quad (21)$$

Hence, we have the RQL for the oscillator in terms of the dimensionless variables x and p which can be the quadratures for a mode of frequency ω of the electromagnetic field,

$$|\sigma^2(x(t)) - \Big(\cos^2(\omega t)\sigma^2(x(0)) - \sin^2(\omega t)\sigma^2(p(0))\Big)| \leq \frac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}, \quad (22)$$

and

$$|\sigma^2(p(t)) - \Big(\sin^2(\omega t)\sigma^2(x(0)) - \cos^2(\omega t)\sigma^2(p(0))\Big)| \leq \frac{1}{2}|\sin(2\omega t)|\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1} \quad (23)$$

The extremal states saturating these RQL may be written in terms of the dimensionless variables x, p for use in optical quadrature measurements,

$$< x'|\psi(0)_\pm >= \Big(\frac{Re\,\eta_\pm}{\pi}\Big)^{1/4} \times \exp\Big(i < p(0) > x') - \frac{\eta_\pm(x' - < x(0) >)^2}{2}\Big), \quad (24)$$

with

$$\eta_\pm = \frac{1}{2\sigma^2(x(0))}[1 \pm i\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}]. \tag{25}$$

The values $\eta = \eta_\pm$ yield the values $\sigma^2(x(t))_\pm$ and $\sigma^2(p(t))_\pm$,

$$\sigma^2(x(t))_\pm - \Big(\cos^2(\omega t)\sigma^2(x(0)) - \sin^2(\omega t)\sigma^2(p(0))\Big)$$
$$= \mp\frac{1}{2}\sin(2\omega t)\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}, \tag{26}$$

and

$$\sigma^2(p(t))_\pm - \Big(\sin^2(\omega t)\sigma^2(x(0)) - \cos^2(\omega t)\sigma^2(p(0))\Big)$$
$$= \pm\frac{1}{2}\sin(2\omega t)\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1} \quad (27)$$

We deduce ,for example, that for the initial state $|\psi(0)_+ >$, in case $\sigma^2(p(0)) - \sigma^2(x(0)) > 0$,

$$\sigma^2(x(t))_+ \leq \sigma^2(x(0)), if$$
$$0 \leq \omega t = \tan^{-1}[\frac{\sqrt{4\sigma^2(x(0))\sigma^2(p(0)) - 1}}{\sigma^2(p(0)) - \sigma^2(x(0))}]; \quad (28)$$

and in the case $\sigma^2(p(0)) - \sigma^2(x(0)) < 0$, we get

$$\sigma^2(x(t))_+ \leq \sigma^2(x(0)),\ if\ 0 \leq \omega t \leq \pi/2. \quad (29)$$

Analogous inequalities are easily obtained for $(\sigma^2(p(t)))_-$ for the initial state $|\psi(0)_- >$. The contractive states for the oscillator thus improve on the Schrödinger coherent states which have constant $\sigma^2(x(t))$, $\sigma^2(p(t))$.

It is easy to rewrite the bounds (22),(23) and extremal states (24) in dimensionless variables in terms of the dimensional X and P

for the oscillator. Thus we have, the RQL for the oscillator,

$$|\sigma^2(X(t)) - \cos^2(\omega t)\sigma^2(X(0)) - \frac{\sin^2(\omega t)}{m^2\omega^2}\sigma^2(P(0))| \leq \frac{|\sin(2\omega t)|}{2m\omega}\sqrt{4\sigma^2(X(0))\sigma^2(P(0)) - \hbar^2}. \tag{30}$$

which shows that in the limit $\omega \to 0$ the RQL for the oscillator (30) yields the RQL for a free mass ,Eq. (12).

For practical realisation of the extremal oscillator states, it is very encouraging that there has been progress in preparing a mechanical oscillator in non-Gaussian quantum states [0] by transfering such states from optical fields onto the oscillator.

Improving the primeval EPR states.

Einstein, Podolsky and Rosen applied the principle of local reality to the "unnormalized quantum state"

$$|q_1 + q_2 = q_0\rangle |p_1 - p_2 = p_0\rangle$$

.

Suppose observers A and B are spacelike separated. If B chooses to measure q_2 she predicts q_1 with certainty (without disturbing that particle since it is spatially separated), and hence q_1 must have physical reality; equally, if she chooses to measure p_2 she predicts p(1) to have reality. A superposition,

$$\int dq_0 dp_0 \phi(q_0)\chi(p_0)|q_1+q_2=q_0\rangle|p_1-p_2=p_0\rangle \tag{31}$$

at time $t = 0$ will constitute a normalizable "nearly EPR" state if the wave packets $\phi(q_0)$ and $\chi(p_0)$ are sufficiently narrow. However , if we start from Gaussian states, the dispersion in $q_1 + q_2$ will then increase rapidly in time thus invalidating the EPR argument; this can be avoided by choosing $\phi(q_0)$ to be a contractive state.

Arthurs-Kelly type realization of accurate monitoring of free mass and harmonic oscillator positions using contractive states.

Two independent commuting meter observables $\hat{x}_1$ and $\hat{x}_2$ (with conjugates $\hat{p}_1$ and $\hat{p}_2$) track respectively the position $\hat{x}$ and momentun $\hat{p}$ of the system. During a short interval T , their interaction Hamiltonian [0],

$$H = K(\hat{x}\hat{p}_1 + \hat{p}\hat{p}_2) \tag{32}$$

is so strong that the free Hamiltonians of the system and apparatus can be neglected.Here K is a const, with $KT = 1$. Denoting by $|\psi(t) >$ the system-apparatus state at time t, we start with the initial state,

$$< x, x_1, p_2|\psi(t = 0) >= \phi(x)\chi_1(x_1)\tilde{\chi}_2(p_2) \tag{33}$$

where $\chi_1(x_1)$, , and $\tilde{\chi}_2(p_2)$ are normalised contractive states,

$$\chi_1(x_1) = \Big(\frac{Re\,\eta_+}{\pi}\Big)^{1/4} \exp\Big(-\frac{\eta_+ x_1^2}{2}\Big),$$
$$\tilde{\chi}_2(p_2) = \Big(\frac{Re\,\eta_+}{4\pi}\Big)^{1/4} \exp\Big(-\frac{\eta_+ p_2^2}{8}\Big), \quad (34)$$

with $\eta_\pm$ given by Eqn. (25). Solving the Schrödinger Eqn. exactly, and taking a Fourier transform, we obtain the final state at time T,

$$< x, x_1, x_2|\psi(T) >= \frac{1}{\sqrt{2\pi}} \chi_{x_1,x_2,\eta_+}(x) \Big\langle \chi_{x_1,x_2,\eta_-}|\phi \Big\rangle, \quad (35)$$

where,

$$\chi_{x_1,x_2,\eta}(x) = \Big(\frac{Re\ \eta}{2\pi}\Big)^{1/4} \exp\Big(ixx_2 - \frac{\eta(x_1-x)^2}{4}\Big). \tag{36}$$

The final probability distribution of the meter variables,

$$P(x_1,x_2) = \frac{1}{2\pi}|\Big\langle\chi_{x_1,x_2,\eta_-}|\phi\Big\rangle|^2 \tag{37}$$

yields the correct system expectation values of position and momentom in the initial state $|\phi >$. After measurement results x_1, x_2, the system is left in the contractive state $|\chi_{x_1,x_2,\eta_+} >$ whose position uncertainty can be much smaller than that of $|\phi>$. The next measurement can be made in the time-window already calculated for the free mass and the oscillator to retain position uncertainty less than that in the state $|\chi_{x_1,x_2,\eta_+}>$. In this way, accurate monitoring of position beating the SQL but respecting the RQL is achieved. Of course the measurement procedure suggested by Ozawa[0] can also be adopted.

Conclusion. I have obtained rigorous quantum limits on the variance $\sigma^2(X(t))$ in terms of $\sigma^2(X(0))$ and $\sigma^2(P(0))$ for arbitrary quantum states of a free mass and oscillator, and also obtained the states which achieve saturation of the limits. I have noted their applications towards accurate monitoring of free mass position ,oscillator position and optical quadratures and given an explicit Arthurs-Kelly like realisation of the corresponding measurement scheme. In the oscillator case the extremal contractive state improves on the Schrödinger coherent states for a well defined time interval, and is likely to be useful in quantum optics and gravitational wave detectors..

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